Scattering problems and asymptotic models at a fixed frequency

Houssem HADDAR
INRIA Saclay & Ecole Polytechnique
houssem.haddar@inria.fr

January 6, 2008

Short Introduction

The solutions to scattering problems in harmonic regime contain an intrinsic length of oscillation λ, called the wavelength. Any numerical method based a discretization of the computational domain should therefore use a mesh size fine enough to capture the variations of the solutions within a period of space λ. Many practical problems require however a stronger constrain on the mesh size due to the presence of a smaller scale δ induced by geometrical fine details or by physical strong contrasts. The cases where $\delta \ll \lambda$ therefore lead to a significant increase in the size of the discrete model and consequently the cost of the numerical scheme.

This course will provide an introduction to some asymptotic techniques that can be used to replace the exact model with approximate ones whose numerical resolution does not suffer from that stronger constrain and whose solution will converge to the exact one as δ goes to zero. In general, one can construct a hierarchy of models with increasing rate of convergence in terms of δ, but also with increasing complexity.

The main focus of this course will be on the theoretical side: understanding the principle of asymptotic expansions, formally deriving approximate models and obtaining error estimates through stability analysis of the problems with respect to the small parameter.

After a general introduction to scattering problems in harmonic regime (we shall restrict ourselves to the scalar problem modeled by the Helmholtz equation, as presented in Chapter 1) we shall investigate three typical asymptotic configurations associated with:

1. The scattering from coated obstacles of thickness $\delta \ll \lambda$. In this case a scaled asymptotic expansion is used to derive so-called Generalized Impedance Boundary Conditions (GIBC) that reproduce the effect of the thin layer without needing to incorporate this thin layer into the computational domain. We shall follow a similar approach as in [9] and in [17].

2. The scattering from strongly absorbing obstacles where the small scale δ is produced by the boundary layer effect due to strong absorption. We shall explain here how scaled asymptotic techniques can be adapted to capture boundary layer effects. They also can be used to derive GIBCs. The treatment of this case is the object of Chapter 2. The extension to Maxwell's problem is summarized in [14] and is treated in [19].

3. The scattering from periodic media where the periodicity scale δ is very small compared to λ. We shall describe here the method of two-scale asymptotic expansion that

can be used the obtain so-called homogenized models. These models only reproduce slow variations of the solutions. The classical references to periodic homogenization theory using two scale asymptotic expansions are [27, 10, 8]. The generalization to non periodic cases is treated in [22].

More complex configurations that couple different small scales (for instance, highly oscillating and/or strongly absorbing thin coatings) need a combination of the above mentioned asymptotic techniques and/or a matching procedure that connect different asymptotic expansions. The technique of matched asymptotic expansions will be briefly described in this context. The analysis of this kind of techniques is left as a reading perspective of this introductory course (see [23] for applications the matching asymptotic method in the case of the Laplace equation).

Contents

Chapter 1

Scattering problems: generalities

1.1 Introduction

This chapter intends to give a quick overview on the physical and mathematical settings of scattering problems in harmonic regime. Classical references on mathematical studies of scattering problems are the Colton-Kress books [13, 12]. We shall adopt here the variational point of view in studying the existence and uniqueness of solutions to the forward problem, which is more suited to the asymptotic analysis introduced in next chapters. Further reading on this approach in the case of scattering problems can be found in [25, 24].

1.1.1 Wave equations

The model problem that will be later introduced can be seen as a simplified version of more complex ones hereafter described. These models are associated with three main categories of wave phenomena.

Acoustic waves: These waves propagate within fluids (gas, liquids, ...) and are generated by pressure variations $p(x,t)$. The simplest mathematical model associated with these waves is the scalar wave equation obtained after linearization of the Euler equations:

$$\frac{\partial^2 p}{\partial t^2} + \gamma(x)\frac{\partial p}{\partial t} - c(x)^2\rho_0(x)\mathrm{div}\,\frac{1}{\rho_0(x)}\nabla p = s(x,t)$$

$c(x)$: the sound speed
$\rho_0(x)$: the reference density
$\gamma(x)$: an absorption coefficient
$s(x,t)$: a source term.
The couple (x,t) denotes the (space, time) coordinates in $\mathbb{R}^3 \times \mathbb{R}$.

Electromagnetic waves: These waves, composed of an electric field and a magnetic field, are generated by electron movements and propagate everywhere. The propagation of these waves is governed by Maxwell's equations. The simplest form of these equations is the one associated with dielectric materials and can be expressed in terms of electric field $\mathbf{E}(x,t)$ (vectorial unknown) as

$$\varepsilon(x)\frac{\partial^2\mathbf{E}}{\partial t^2}+\sigma(x)\frac{\partial\mathbf{E}}{\partial t}+\operatorname{curl}\frac{1}{\mu(x)}\operatorname{curl}\mathbf{E}=\mathbf{J}(x,t)$$

$\varepsilon(x)$: electric permittivity
$\mu(x)$: magnetic permeability
$\sigma(x)$: electric conductivity
$\mathbf{J}(x,t)$: a source term

Elastic waves: These waves propagate within solids and are generated by displacement variations $\mathbf{u}(x,t)$ (vectorial field). The simplest mathematical model is the linearized elastodynamic equation

$$\rho(x)\frac{\partial^2\mathbf{u}}{\partial t^2}+\gamma(x)\frac{\partial\mathbf{u}}{\partial t}-\operatorname{div}\sigma(\mathbf{u})=\mathbf{f}(x,t)$$

$\rho(x)$: density
$\gamma(x)$: an absorption coefficient
$\sigma(\mathbf{u})$: the stress tensor whose linear dependence in terms of $\mathbf{u}$ is given by the Hooke law. For isotropic materials

$$\sigma(\mathbf{u})=\lambda\operatorname{tr}(\epsilon)\delta_{i,j}+2\mu\,\epsilon_{i,j}$$

where $\epsilon_{i,j}:=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$.
λ, μ : the Lame coefficients
$\mathbf{f}(x,t)$: a source term

1.1.2 The scalar model problem

Our model problem is the scalar wave equation. The unknown $u(x,t)$ (that plays for instance the role of the pressure variation) satisfies

$$\frac{1}{c(x)^2}\frac{\partial^2 u}{\partial t^2}+\frac{\gamma(x)}{c(x)^2}\frac{\partial u}{\partial t}-\Delta u=\tilde{s}(x,t), \tag{1.1}$$

where (x,t) denotes the (space, time) coordinates in $\mathbb{R}^N\times\mathbb{R}$ with N denoting the space dimension ($N=1,2$ or 3). The coefficients $c(x)$ and $\gamma(x)$ are non negative quantities and are equal to those associated with the free space outside a bounded region B:

$$c(x)=c_0>0 \quad\text{and}\quad \gamma(x)=0 \qquad x\notin B.$$

The domain B may also contain sound-soft or sound-hard obstacles that occupy a subdomain D. On the boundary of a connected component of D the solution u satisfies either of the boundary conditions

$$u = 0 \text{ or } \frac{\partial u}{\partial n} = 0 \text{ on } \partial D \tag{1.2}$$

where n denotes the unitary normal to ∂D directed to the interior of D.

Exercise 1.1

- *Prove that for elastic waves, $u = \operatorname{div} \mathbf{u}$ satisfies a similar equation as (1.1).*
- *Prove that for electromagnetic waves, if the medium and the solutions are invariant along x_3 direction then $u = E_3$ also satisfies a similar equation as (1.1).*

Incident waves: Incident waves $u_i(x,t)$ are defined (in general) as solutions to the problem when the medium does not contain any inhomogeneities or obstacles. They satisfy

$$\frac{1}{c_0^2}\frac{\partial^2 u_i}{\partial t^2} - \Delta u_i = \tilde{s}(x,t). \tag{1.3}$$

Scattered wave: Denoted by $u_s(x,t)$, this field is defined by $u_s(x,t) := u(x,t) - u_i(x,t)$. It therefore corresponds to the wave generated by any perturbation of the free space. Taking the difference between (1.1) (1.3) one gets

$$\frac{1}{c(x)^2}\frac{\partial^2 u_s}{\partial t^2} + \frac{\gamma(x)}{c(x)^2}\frac{\partial u_s}{\partial t} - \Delta u_s = -\left(\frac{1}{c(x)^2} - \frac{1}{c_0^2}\right)\frac{\partial^2 u_i}{\partial t^2} - \frac{\gamma(x)}{c(x)^2}\frac{\partial u_i}{\partial t}. \tag{1.4}$$

We observe that the diffracted field satisfies a wave equation with a source term generated by the incident field and whose support is contained into B.

We shall consider problems with an established harmonic regime, i.e. when the sources and the solutions have harmonic dependence with respect to t. More precisely we assume that

$$\begin{aligned}
\tilde{s}(x,t) &= \Re e(s(x)\exp(-i\omega t)),\\
u(x,t) &= \Re e(u(x)\exp(-i\omega t)),\\
u_i(x,t) &= \Re e(u_i(x)\exp(-i\omega t)),\\
u_s(x,t) &= \Re e(u_s(x)\exp(-i\omega t)),
\end{aligned}$$

where $\omega > 0$ is the pulsation that we shall (abusively) also refer to as the frequency. Formally, the new unknown $u(x)$ can be interpreted as the Fourier transform of $t \mapsto u(x,t)$ evaluated at the frequency ω. Notice that the unknowns are now complex valued functions and that getting the "physical" solution $u(x,t)$ requires both the real and imaginary parts of $u(x)$.

Substituting in (1.1), (1.3) and (1.4) we observe that the incident field $u_i(x)$ satisfies (the Helmholtz equation),

$$-\kappa^2 u_i - \Delta u_i = s(x) \qquad \text{in } \mathbb{R}^N \setminus \overline{D} \tag{1.5}$$

where $\kappa := \omega/c_0$ is the wavenumber, while the total field u satisfies

$$-\kappa^2 q(x) u - \Delta u = s(x) \qquad \text{in } \mathbb{R}^N \setminus \overline{D} \tag{1.6}$$

where

$$q(x) := \frac{c_0^2}{c(x)^2}\left(1 + i\frac{\gamma(x)}{\omega}\right) \tag{1.7}$$

is referred to as the medium index. Notice that $n = 1$ outside B and that $\Im m(n) \geq 0$. The diffracted field satisfies

$$-\kappa^2 q(x) u_s - \Delta u_s = -\kappa^2(1 - q(x))u_i \qquad \text{in } \mathbb{R}^N \setminus \overline{D}, \tag{1.8}$$

These equations are supplemented with the boundary conditions (1.2).

1.1.3 On the radiation condition

The above equations are not sufficient to uniquely determine the scattered wave (assuming that u_i is given). To guarantee the uniqueness of solutions one needs to supplement these equations with a criterion that selects physical solutions, namely the Sömmerfeld radiation condition. In order to understand the origin and physical meaning of this radiation condition we propose to consider the simplified problem where we seek the wave generated in the free space by a source term $f(x)$ of compact support. We are then interested in physical solutions u_s satisfying

$$-\kappa^2 u_s - \Delta u_s = f(x) \qquad \text{in } \mathbb{R}^N. \tag{1.9}$$

One possible way to compute those solutions is to first define the "physical" fundamental solution G_N satisfying

$$-\kappa^2 G_N - \Delta G_N = \delta \text{ in } \mathbb{R}^N. \tag{1.10}$$

Then the "physical" solution of (1.9) is simply obtained by convolution:

$$u_s(x) = (G_N * f)(x) \quad x \in \mathbb{R}^N$$

Exercise 1.2 *Prove that the (spherically invariant) solutions to (1.10) are given by*

$$\tilde{G}_q(x) = \alpha G_q(x) + (1-\alpha)\overline{G_N}(x) \tag{1.11}$$

$\alpha \in \mathbb{C}$, *where*

$$G_2(x) = \frac{i}{4} H_0^{(1)}(\kappa|x|), \qquad G_3(x) = \frac{e^{i\kappa|x|}}{4\pi|x|} \tag{1.12}$$

with $H_0^{(1)}$ denoting the cylindrical Hankel function of the first kind of order zero[1]. *Verify that $G_N \in L^1_{loc}(\mathbb{R}^3)$.*

Let us set

$$\Phi_N(x,y) := G_N(x-y), \;\; x, y \in \mathbb{R}, \; x \neq y$$

so that we can express the solutions to (1.9) in the form

$$u_s(x) = \alpha \int_B \Phi(x,y) f(y) dy + (1-\alpha) \int_B \bar{\Phi}(x,y) f(y) dy = \alpha u_s^1(x) + (1-\alpha) u_s^2(x)$$

where B is a bounded domain containing the support of f.

Exercise 1.3 *Prove that*

$$\Phi_N(x,y) = \frac{e^{i\kappa|x|}}{C_N |x|^{(N-1)/2}} \left(e^{-i\kappa \hat{x}\cdot y} + O\left(\frac{1}{|x|}\right)\right) \tag{1.13}$$

uniformly with respect to $\hat{x} := x/|x|$ and with respect to y in B, where C_N is a constant. Indication: use the identity $|x-y| = |x|\sqrt{1 - 2\hat{x}\cdot \frac{y}{|x|} + \frac{|y|^2}{|x|^2}} = |x|(1 - \hat{x}\cdot\frac{y}{|x|} + O(\frac{1}{|x|^2}))$.

The expansion (1.13) shows that

$$\begin{aligned} u_s^1(x) &= \tfrac{e^{i\kappa|x|}}{C_N|x|^{(N-1)/2}} \left(u_\infty^1(\hat{x}) + O\left(\tfrac{1}{|x|}\right)\right) \\ u_s^2(x) &= \tfrac{e^{i\kappa|x|}}{C_N|x|^{(N-1)/2}} \left(u_\infty^2(\hat{x}) + O\left(\tfrac{1}{|x|}\right)\right) \end{aligned} \tag{1.14}$$

uniformly with respect to $\hat{x}$, where

$$u_\infty^1(\hat{x}) = \int_D e^{-i\kappa\hat{x}\cdot y} f(y) dy \quad \text{and} \quad u_\infty^1(\hat{x}) = \int_D e^{i\kappa\hat{x}\cdot y} f(y) dy.$$

Coming back to the time-domain solutions, we observe that

$$\begin{aligned} u_s^1(x,t) &\sim \Re e \left\{ u_\infty^1(\hat{x}) \frac{e^{i\kappa(|x|-c_0 t)}}{C_N|x|^{(N-1)/2}} \right\}, \\ u_s^2(x,t) &\sim \Re e \left\{ u_\infty^2(\hat{x}) \frac{e^{-i\kappa(|x|+c_0 t)}}{C_N|x|^{(N-1)/2}} \right\}, \end{aligned} \tag{1.15}$$

for large $|x|$.

The asymptotic behavior of u_s^1 corresponds to an outgoing spherical wave (positive phase velocity) while the one associated with u_s^2 corresponds to an incoming spherical wave (negative phase velocity). Hence only $u_s^1(x,t)$ is a physically acceptable solution and

$$u_s(x) = u_s^1(x) = \int_D \Phi(x,y) f(y) dy.$$

[1] We refer to the first chapter of [13] for a quick and self-contained presentation of spherical Bessel and Hankel functions. See also the Wikipedia web page `http://en.wikipedia.org/wiki/Bessel_functions` and the classical reference [29]

Exercise 1.4 *Prove that if we have chosen a temporal harmonic dependency like* $\exp(i\omega t)$ *then* u_s^2 *will correspond to physical solution.*

We shall now derive the radiation condition that will enable us to select u_s^1 and withdraw other solutions. Similarly to (1.13) one easily verify that

$$\nabla\Phi_N(x,y) = \frac{i\kappa e^{i\kappa|x|}}{C_N|x|^{(N-1)/2}}\left(e^{-i\kappa\hat{x}\cdot y}\hat{x} + O\left(\frac{1}{|x|}\right)\right) \tag{1.16}$$

uniformly with respect to $\hat{x} := x/|x|$ and with respect to y in B. Therefore

$$\begin{aligned}\partial_r u_s^1(x) &= \frac{i\kappa e^{i\kappa|x|}}{C_N|x|^{(N-1)/2}}\left(u_\infty^1(\hat{x}) + O\left(\frac{1}{|x|}\right)\right)\\ \partial_r u_s^2(x) &= -\frac{i\kappa e^{i\kappa|x|}}{C_N|x|^{(N-1)/2}}\left(u_\infty^2(\hat{x}) + O\left(\frac{1}{|x|}\right)\right)\end{aligned} \tag{1.17}$$

uniformly with respect to $\hat{x}$. Consequently

$$|x|^{(N-1)/2}(\partial_r u_s^1(x) - i\kappa u_s^1(x)) = O(1/|x|)$$

while

$$(\partial_r u_s^2(x) - i\kappa u_s^2(x)) = -2i\kappa e^{i\kappa|x|}\left(u_\infty^2(\hat{x}) + O\left(\frac{1}{|x|}\right)\right).$$

It seems therefore natural to require that "physical" solutions should satisfy the radiation condition

$$\lim_{r\to\infty}|x|^{(N-1)/2}(\partial_r u_s(x) - i\kappa u_s(x)) = 0 \tag{1.18}$$

uniformly with respect to $\hat{x}$. This is the so-called Sömmerfeld radiation condition. As later shown, this condition is sufficient to guarantee uniqueness of the scattering problem solutions.

Remark 1.1

- *The weaker formulation of (1.18)*

$$\lim_{R\to\infty}\int_{S_R}|\partial_r u_s - i\kappa u_s|^2\,ds = 0 \tag{1.19}$$

where S_R *denotes the sphere of radius* R *in* $\mathbb{R}^N$, *is also sufficient guarantee uniqueness of the scattering problem solutions. This is the form we shall later use.*

- *To a time dependency in* $\exp(i\omega t)$ *one should associate the radiation condition*

$$\lim_{R\to\infty}\int_{S_R}|\partial_r u_s + i\kappa u_s|^2\,ds = 0$$

Remark 1.2 (Limiting absorption principle) *Let* $\epsilon > 0$ *be a small parameter.*

- *The "physical" solution u_s^1 can also be defined as the $L^2_{\text{loc}}(\mathbb{R}^N)$ limit as $\epsilon \to 0^+$ of the unique $L^2(\mathbb{R}^N)$ solution to (1.9) when κ^2 is replaced with $\kappa^2(1+i\epsilon)$, which corresponds to adding a small absorption coefficient $\gamma = \omega\epsilon$.*

- *The solution u_s^2 is obtained by a limiting absorption process when κ^2 is replaced with $\kappa^2(1-i\epsilon)$. However the latter correspond to non physical absorption since in that case $\gamma = -\omega\epsilon < 0$.*

1.1.4 Some mathematical properties of radiating solutions

We summarize here some of the key properties of radiating solutions. These properties are left as an exercise to the reader. The details of the proof can be found in [13].

Definition 1.1 *Let B be a bounded set of $\mathbb{R}^N$. We denote by radiating solution to the Helmholtz equation in $\mathbb{R}^3 \setminus B$ any function $u \in H^2_{\text{loc}}(\mathbb{R}^N \setminus B)$ satisfying*

$$\Delta u + \kappa^2 u = 0 \ \ in \ \ \mathbb{R}^3 \setminus D.$$

together with the radiation (1.19), namely

$$\lim_{R\to\infty} \int_{S_R} |\partial_r u - i\kappa u|^2 \, ds = 0.$$

Notice that $x \mapsto \Phi(x, y)$ is a radiating solution to the Helmholtz equation in $\mathbb{R}^3 \setminus \{y\}$. Let B be a bounded regular domain of $\mathbb{R}^N$ and u be a radiating solution to the Helmholtz equation in $\mathbb{R}^3 \setminus B$:

1. There exists a constant c independent of R such that $\displaystyle\int_{S_R} |u|^2 \, ds \leq c$ for all R large enough.

2. Let v be another radiating solution to the Helmholtz equation in $\mathbb{R}^3 \setminus B$ then
$$\int_{\partial B} (u\partial_n v - v\partial_n u) \, ds = 0.$$

3. Let $x \in B$ then
$$\int_{\partial B} (u\partial_n \Phi(\cdot, x) - \Phi(\cdot, x)\partial_n u) ds = 0.$$

4. Let $x \notin \overline{D}$ then
$$u(x) = \int_{\partial B} (u\partial_n \Phi(\cdot, x) - \Phi(\cdot, x)\partial_n u) ds.$$

5. There exists an analytic function u_∞ defined on the unit sphere S^{N-1} of $\mathbb{R}^N$, called the far-field, such that

$$u(x) = \frac{e^{i\kappa|x|}}{C_N|x|^{(N-1)/2}}\left(u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right).$$

6. The far-field is given by

$$u_\infty(\hat{x}) = \int_{\partial B}(u(y)\partial_n e^{-i\kappa\hat{x}\cdot y} - e^{-i\kappa\hat{x}\cdot y}\partial_n u(y)\, ds(y).$$

7. As a complementary statement to the first point one has

$$\lim_{R\to\infty}\int_{S_R}|u|^2\, ds = \int_{S^{N-1}}\left|\frac{u_\infty(\hat{x})}{C_N}\right|^2\, ds(\hat{x}).$$

1.2 Study of the scattering problem

Let B a bounded set of $\mathbb{R}^N$ and $D \subset B$ an open set with Lipschitz boundary. Let q be a complex bounded function of $\mathbb{R}^N \setminus D$ such that

$$\Im m(q(x)) \geq 0 \text{ for } x \notin D \quad \text{and} \quad q(x) = 1 \text{ for } x \notin B. \tag{1.20}$$

Let Γ_D be the (open) part of ∂D where the Dirichlet condition holds and $\Gamma_N := \partial D \setminus \overline{\Gamma}_D$ the part where the Neumann boundary condition holds. We shall introduce the notation for $m > 0$, and a set $\mathcal{O}$ containing D,

$$\mathcal{H}^m(\mathcal{O}) := \{u \in H^1_{\text{loc}}(\overline{\mathcal{O}} \setminus D) \cap H^m_{\text{loc}}(\mathcal{O} \setminus \overline{D}); \quad u_{|\Gamma_D} = 0\} \tag{1.21}$$

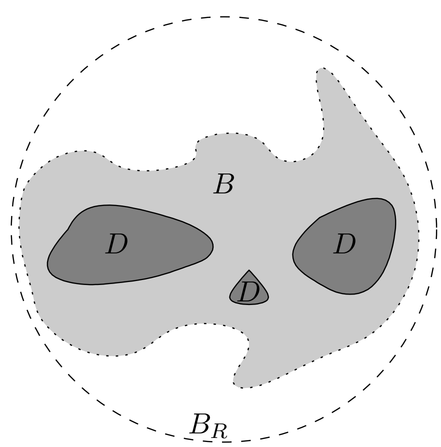


Figure 1.1: Description of the problem geometry

1.2.1 Uniqueness

We shall first prove that the radiation condition is sufficient to ensure the uniqueness of the solutions. This uniqueness result will also be used to prove existence of solutions.

Theorem 1.1 *Any function $u \in \mathcal{H}^2(\mathbb{R}^N)$ satisfying*

$$\begin{cases} \Delta u + \kappa^2 q u = 0 \quad \text{in } \mathbb{R}^N \setminus \overline{D}, \\ \partial u/\partial n = 0 \quad \text{on } \Gamma_N, \\ \lim_{R\to\infty} \int_{S_R} |\partial_r u - i\kappa u|^2 \, ds = 0, \end{cases} \tag{1.22}$$

must be identically zero in $\mathbb{R}^N \setminus D$.

The first ingredient to the proof is the Rellich Lemma. From now on, B_R will denote the open ball of radius R.

Lemma 1.1 (Rellich) *Let $a > 0$ and $u \in H^2_{loc}(\mathbb{R}^N \setminus B_a)$ satisfying $\Delta u + \kappa^2 u = 0$ in $(\mathbb{R}^N \setminus B_a)$. Then, either of the following holds*

- *the function u is identically 0 for $|x| \geq a$.*
- *there exists a constant $c > 0$ such that $\displaystyle\int_{a<|x|<R} |u|^2 dx > cR$ for all $R > a$.*

This Lemma indicates that any non trivial solution to the Helmholtz equation cannot decay faster than $1/|x|$ for large $|x|$. A proof of this lemma can be found in [13].

Exercise 1.5 *Prove the Rellich lemma when u is spherically symmetric.*

The Rellich Lemma can be used in different ways and the one that we shall later apply is the following immediate corollary.

Corollary 1.1 *Let $a > 0$ and $u \in H^2_{loc}(\mathbb{R}^N \setminus B_a)$ satisfying $\Delta u + \kappa^2 u = 0$ in $\mathbb{R}^N \setminus B_a$.*

If $\displaystyle\lim_{R\to\infty} \int_{S_R} |u|^2 \, ds = 0$, then u is identically 0 for $|x| \geq a$.

Remark 1.3 *Another immediate consequence of the Rellich Lemma is obtained after combining Corollary 1.1 with the statement number 7 of Section 1.1.4: For any radiating solution u to the Helmholtz equation in $\mathbb{R}^N \setminus B_a$, $u_\infty = 0$ implies $u = 0$ in $\mathbb{R}^3 \setminus B_a$.*

The second ingredient to the uniqueness proof is the unique continuation principle.

Lemma 1.2 (Unique continuation principle) *Let $\mathcal{O}$ be an open set and $u \in H^2(\mathcal{O})$ such that there exists a constant C for which*

$$|\Delta u(x)| \leq C(|\nabla u(x)| + |u(x)|) \quad \text{for a.e. } x \in \mathcal{O}.$$

If u vanishes a.e. in a neighborhood of a point $x_0 \in \mathcal{O}$ then u vanishes a.e. in $\mathcal{O}$.

Proof of Theorem 1.1. Let $R > 0$ be such that $B \subset B_R$. Multiply the first equation of (1.22) by $\bar{u}$ then integrate over B_R and apply the Green formula to get

$$-\int_{B_R\setminus D} |\nabla u|^2 dx + \kappa^2 \int_{B_R\setminus D} q|u|^2 dx + \int_{S_R} \partial_r u \, \bar{u} \, ds = 0.$$

Hence

$$\Im m \int_{S_R} \partial_r u \, \bar{u} \, ds = -\kappa^2 \int_{B_R\setminus D} \Im m(q)|u|^2 dx \leq 0.$$

Since $|\partial_r u - i\kappa u|^2 = |\partial_r u|^2 + \kappa^2|u|^2 - 2k\Im m(\partial_r u \, \bar{u})$, one deduces

$$\int_{S_R} |u|^2 ds \leq \frac{1}{\kappa^2} \int_{S_R} |\partial_r u - i\kappa u|^2 ds \to 0$$

as $R \to \infty$. Corollary 1.1 therefore implies that $u = 0$ in $\mathbb{R}^N \setminus B_a$ where B_a is a ball containing B. Now observe that

$$|\Delta u(x)| \leq \kappa^2 |q|_\infty |u(x)| \quad \text{for a.e. } x \in \mathbb{R}^N \setminus \overline{D}.$$

Hence according to Lemma 1.2, u must vanish in $\mathbb{R}^N \setminus \overline{D}$. □

1.2.2 Existence of solutions: variational approach

The construction of solutions using a variational approach will be done in three steps:

- Introduce the so-called Dirichlet-to-Neumann (DtN) map for the exterior problem.
- Use the DtN map to write an equivalent formulation of the scattering problem in a bounded domain.
- Write the variational formulation associated with the problem in a bounded domain and apply the Fredholm alternative to prove existence and uniqueness of solutions.

Definition 1.2 (DtN map for the exterior problem) *Let R be a given positive real. We define the DtN map T_R as*

$$\begin{array}{rccc} T_R \; : & H^{\frac{1}{2}}(S_R) & \longrightarrow & H^{-\frac{1}{2}}(S_R) \\ & \varphi & \longmapsto & T_R(\varphi) = \partial_r w_{|S_R} \end{array}$$

where $w \in H^1_{\mathrm{loc}}(\mathbb{R}^N \setminus B_R) \cap H^2_{\mathrm{loc}}(\mathbb{R}^N \setminus \overline{B_R})$ is the unique solution to

$$\begin{cases} \Delta w + \kappa^2 w = 0 \quad \text{in } \mathbb{R}^N \setminus \overline{B}_R, \\ w = \varphi \quad \text{on } S_R, \\ \displaystyle\lim_{R\to\infty} \int_{S_R} |\partial_r w - i\kappa w|^2 \, ds = 0. \end{cases} \tag{1.23}$$

Let us denote by $\langle\, ,\, \rangle_{S_R}$ the duality product between $H^{-\frac{1}{2}}(S_R)$ and $H^{\frac{1}{2}}(S_R)$ that coincides with $L^2(S_R)$ scalar product for regular functions.

Theorem 1.2 *The DtN map T_R is well defined and is a continuous map from $H^{\frac{1}{2}}(S_R)$ into $H^{-\frac{1}{2}}(S_R)$. Moreover,*

$$\Im m \,\langle T_R(\varphi),\, \varphi\rangle_{S_R} \geq 0 \quad \textit{and} \quad \Re e \,\langle T_R(\varphi),\, \varphi\rangle_{S_R} \leq 0 \quad \textit{for all } \varphi \in H^{\frac{1}{2}}(S_R).$$

The proof of this theorem can be found in [25]. It is based on the construction of solutions to (1.23) by using the separation of variables (and therefore is dependent on the space dimension). For $\varphi \in L^2(S_R)$ we define

$$\hat{\varphi}_\ell(\hat{x}) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \varphi(R\hat{y}) e^{-i\ell\theta(\hat{y})} d\theta(\hat{y})\; e^{i\ell\theta(\hat{x})} & \text{for } N = 2, \\ \displaystyle\sum_{-\ell}^{\ell} \int_{S^{N-1}} \varphi(R\hat{x}) Y_{-\ell}^m(\hat{x}) ds(\hat{x})\; Y_\ell^m(\hat{x}) & \text{for } N = 3, \end{cases}$$

where $\theta(\hat{x})$ denotes the cylindrical angle associated with $\hat{x}$ and where Y_ℓ^m, $m = -\ell, \ldots, \ell$ denote the spherical harmonics of order ℓ.

Lemma 1.3 *The solution to (1.23) is given by*

$$w(x) = \sum_{-\infty}^{\infty} \psi_\ell(k|x|)/\psi_\ell(kR)\; \hat{\varphi}_\ell(\hat{x}) \quad \textit{for } |x| \geq R,$$

and the DtN map is given by

$$T_R(\varphi)(R\hat{x}) = \sum_{-\infty}^{\infty} k\, \psi_\ell'(kR)/\psi_\ell(kR)\; \hat{\varphi}_\ell(\hat{x}) \quad \textit{for } \hat{x} \in S^{N-1},$$

where ψ_ℓ denotes the Hankel function (cylindrical for $N = 2$ and spherical for $N = 3$) of the first kind of order ℓ.

Exercise 1.6 *Show that property $\Im m \,\langle T_R(\varphi),\, \varphi\rangle_{S_R} \geq 0$ is independent from the explicit expression of T_R and only depends on the radiation condition.*

Formulation of the scattering problem in a bounded domain

Let $R > 0$ large enough so that B_R contains B. The data of the scattering problem will be some incident field u_i satisfying

$$\Delta u_i + \kappa^2 u_i = s \quad \text{in } \mathbb{R}^N$$

where s is some possible source term. To simplify the exposure we shall assume that $s \in L^2(\mathbb{R}^N)$ (see remark below) and therefore $u_i \in H^2_{\text{loc}}(\mathbb{R}^N)$.

The scattering problem can then be formulated as finding $u \in \mathcal{H}^1(\mathbb{R}^N)$ satisfying

$$\begin{cases} \Delta u + \kappa^2 q u = s \quad \text{ in } \mathbb{R}^N \setminus \overline{D}, \\ \partial u/\partial n = 0 \quad \text{ on } \Gamma_N, \\ u = u_i + u_s \quad \text{ in } \mathbb{R}^N \setminus \overline{D}, \\ \lim_{R\to\infty} \int_{S_R} |\partial_r u_s - i\kappa u_s|^2 \, ds = 0. \end{cases} \tag{1.24}$$

We observe that if u is a solution to (1.24) then $u_{s|\mathbb{R}^N \setminus B_R}$ is a solution to (1.23) with $\varphi = u_{s|S_R}$. Therefore

$$\partial_r u_{s|S_R} = T_R(u_{s|S_R}),$$

and $\tilde{u} := u_{|B_R} \in \mathcal{H}^1(B_R)$ is a solution of the problem

$$\begin{cases} \Delta \tilde{u} + \kappa^2 q \tilde{u} = s \quad \text{ in } B_R \setminus \overline{D}, \\ \partial \tilde{u}/\partial n = 0 \quad \text{ on } \Gamma_N, \\ \partial_r \tilde{u} - T_R(\tilde{u}) = g \quad \text{ on } S_R, \end{cases} \tag{1.25}$$

where

$$g := (\partial_r u_i)_{|S_R}) - T_R(u_{i|S_R}). \tag{1.26}$$

Conversely, let $\tilde{u} \in \mathcal{H}^1(\overline{B}_R)$ satisfying (1.25) and define $\tilde{u}_s$ to be the solution of (1.23) with $\varphi = (\tilde{u} - u_i)_{|S_R}$. One easily verifies that the function

$$u = \begin{cases} \tilde{u} \quad \text{ in } B_R \\ \tilde{u}_s + u_i \quad \text{ in } \mathbb{R}^N \setminus B_R \end{cases}$$

is in $\mathcal{H}^1(\mathbb{R}^N)$ and satisfies (1.24). The fact that $u \in \mathcal{H}^2(\mathbb{R}^N)$ follows from standard elliptic regularity. We therefore have shown the equivalence between the two formulations: (1.24) and (1.25), of the scattering problem. Consequently, as an immediate consequence of the uniqueness result of Theorem 1.1, we can state

Theorem 1.3 *Any function $u \in \mathcal{H}^1(B_R)$ satisfying*

$$\begin{cases} \Delta u + \kappa^2 q u = 0 \quad \textit{ in } B_R \setminus \overline{D}, \\ \partial u/\partial n = 0 \quad \textit{ on } \Gamma_N, \\ \partial_r u - T_R(u_{|S_R}) = 0 \quad \textit{ on } S_R, \end{cases}$$

must be identically zero in $B_R \setminus D$.

Variational study of the problem in a bounded domain

We shall now study the existence of solutions $u \in \mathcal{H}^1(B_R)$ satisfying

$$\begin{cases} \Delta u + \kappa^2 q u = f \quad \text{in } B_R \setminus \overline{D}, \\ \partial u/\partial n = 0 \quad \text{on } \Gamma_N, \\ \partial_r u - T_R(u_{|S_R}) = g \quad \text{on } S_R, \end{cases} \tag{1.27}$$

where $f \in L^2(B_R \setminus D)$ and $g \in H^{-\frac{1}{2}}(S_R)$ are some given data. The equivalent variational formulation of this problem consists into seeking $u \in \mathcal{H}^1(B_R)$ such that

$$-\int_{B_R\setminus D} \nabla u \nabla \bar{v}\, dx + \kappa^2 \int_{B_R\setminus D} q\, u\, \bar{v}\, dx + \left\langle T_R(u_{|S_R}), v \right\rangle_{S_R} = \int_{B_R\setminus D} f\, \bar{v}\, dx - \langle g, v\rangle_{S_R}$$

for all $v \in \mathcal{H}^1(B_R)$. This formulation can be written of the form

$$\alpha(u,v) + \beta(u,v) = l(v) \text{ for all } v \in \mathcal{H}^1(B_R), \tag{1.28}$$

where we have set

$$\alpha(u,v) = \int_{B_R\setminus D} \nabla u \nabla \bar{v}\, dx + \int_{B_R\setminus D} u\, \bar{v}\, dx - \left\langle T_R(u_{|S_R}), v \right\rangle_{S_R}, \tag{1.29}$$

$$\beta(u,v) = -\int_{B_R\setminus D} (\kappa^2 q + 1)\, u\, \bar{v}\, dx, \tag{1.30}$$

$$l(v) = -\int_{B_R\setminus D} f\, \bar{v}\, dx + \langle g, v\rangle_{S_R}. \tag{1.31}$$

It is easily verified that:

- The sesquilinear form α is a continuous and coercive on $\mathcal{H}^1(B_R)\times\mathcal{H}^1(B_R)$. Therefore, using the Lax-Milgram theorem, the operator $A : \mathcal{H}^1(B_R) \to \mathcal{H}^1(B_R)$ defined by

$$(A(u), v)_{\mathcal{H}^1(B_R)} = \alpha(u,v)\ \forall v \in \mathcal{H}^1(B_R) \tag{1.32}$$

  is an isomorphism.

- The sesquilinear form β is a continuous on $L^2(B_R) \times \mathcal{H}^1(B_R)$. Therefore, due to the compact embedding of $\mathcal{H}^1(B_R)$ into $L^2(B_R \setminus D)$ the operator $B : \mathcal{H}^1(B_R) \to \mathcal{H}^1(B_R)$ defined by

$$(B(u), v)_{\mathcal{H}^1(B_R)} = \beta(u,v)\ \forall v \in \mathcal{H}^1(B_R) \tag{1.33}$$

  is compact.

- The antilinear l is continuous on $\mathcal{H}^1(B_R)$

Existence and uniqueness of solutions to (1.28) is therefore equivalent to the invertability of the operator $A - B$. According to the Fredholm alternative, the continuous invertibility is equivalent to the injectivity of this operator. Since the latter is a direct consequence of Theorem 1.3, we are now in position to state:

Theorem 1.4 *The problem (1.27) has a unique solution $u \in \mathcal{H}^1(B_R)$. Moreover, there exists a constant C independent of u and f and g such that*

$$\|u\|_{H^1(B_R\setminus D)} \leq C \left(\|f\|_{L^2(B_R\setminus D)} + \|g\|_{H^{-\frac{1}{2}}(S_R)} \right).$$

Exercise 1.7 *We propose as an exercise slight extensions of the previous theorem when more general conditions, namely impedance boundary conditions, hold on Γ_N.*

- *Prove Theorem 1.4 when the Neumann condition is replaced by*

  $$\partial_n u - Z\, u = g \qquad \text{on } \Gamma_N, \tag{1.34}$$

  where Z is a bounded function on Γ such that

  $$\mathcal{R}e(Z) \leq 0 \text{ and } \mathcal{I}m(Z) \geq 0$$

  and where n denotes the normal to Γ directed to the interior of D

- *Prove the same result when only*

  $$\mathcal{I}m(Z) \geq z_0 > 0$$

  where z_0 is a constant.

Chapter 2

Generalized Impedance Boundary Conditions for Strongly Absorbing Obstacles: The scalar Case

2.1 Introduction

The concept of Generalized Impedance Boundary condition ($GIBC$) is now a rather classical notion in the mathematical modeling of wave propagation phenomena (see for instance [20] and [28]). Such tool is particularly used in electromagnetism for diffraction problems, in the time harmonic regime, by obstacles that are partially or totally penetrable. The general idea is, as soon as it is possible and desirable, to replace the use of an "exact model" inside (the penetrable part of) the obstacle by approximate boundary conditions (also called equivalent or effective conditions) on the boundary of the scatterer. This idea is pertinent in practice when the boundary condition appears to be easy to handle from the numerical point of view, which would be the case if it can be expressed with the help of differential operators. The same type of idea, even though the purpose was different, led to the construction of local absorbing boundary conditions for the wave equation ([15, 11]) or more recently to the construction of On Surface Radiation Conditions ([4, 3]) for pure exterior problems.

The diffraction problem of electromagnetic waves by perfectly conducting obstacles coated with a thin layer of dielectric material is a prototype problem for the use of impedance conditions. Indeed, due to the small (typically with respect to the wavelength) thickness of the coating, the effect of the layer on the exterior medium is, as a first approximation, local (see for instance [28, 20, 16, 9, 7, 6, 1, 2])

Another application, the one we have in mind here, is the diffraction of waves by strongly absorbing obstacles, typically highly conducting bodies in electromagnetism. This time, it is a well-known physical phenomenon, the so-called skin effect, that creates a "thin layer" effect . The conductivity limits the penetrable region to a boundary layer whose depth is inversely proportional to the conductivity of the medium. Then, here again, the effect of the obstacle is, as a first approximation, local. The numerical results presented in Figures 2.2-2.5 of section 2.2.3 illustrate this skin effect phenomena.

As a matter of fact, the research on effective boundary conditions for highly absorbing obstacles began with Leontovich before the apparition of computers and the development of numerical methods. He proposed an impedance boundary condition, that is nowadays known as the Leontovich boundary condition (and that correspond with the condition of order 1 in this paper). This condition only "sees" locally the tangent plane to the frontier. Later, Rytov [26, 28] proposed an extension of the Leontovich condition which was already based on the principle of an asymptotic expansion. More recently, Antoine-Barucq-Vernhet [5] proposed a new derivation of impedance boundary conditions based on the technique of expansion of pseudo-differential operators (following there the original ideas of Engquist-Majda [15] for absorbing boundary conditions).

We shall present in this chapter the approach proposed in [18] in deriving and analyzing $GIBC$s for the scattering of waves by highly absorbing obstacles. This approach based on scaled asymptotic expansion has the advantage of leading to sharp error estimates for the approximate solutions.It permits in particular to give a sense to the *order* of a given $GIBC$, a notion whose meaning is not always clear (at least not always the same) in the literature (it is sometimes related to the order of the differential operators involved in the condition, sometimes linked to the truncation order of some Taylor expansion,...): a GIBC will be of order k if it provides an $O(\varepsilon^{k+1})$ error. A point deserves to be emphasized in this introduction: for a given order k there is not uniqueness of the $GIBC$. We shall illustrate this fact here by presenting several $GIBC$s of order 2 and 3 ; for the same order, different $GIBC$s only differ by (maybe important) other features such as their adaptation to a given numerical methods.

2.2 Model settings

We shall adopt here the same notation as in Section 1.2. We assume that the medium B contains a strongly absorbing inclusion characterized by a large value of the imaginary part of the inclusion index. Let Ω be the domain occupied by this inclusion. We assume that Ω is simply connected with a regular (C^∞) boundary Γ. Without affecting the generality of our analysis we assume that $D = \emptyset$.
The wave speed c_{in} inside Ω is assumed to be constant and the absorption coefficient γ^ε of the medium is assumed to be of the form

$$\gamma^\varepsilon = \begin{cases} \gamma(x) & x \notin \Omega, \\ c_{\text{in}}/(k_{\text{in}}\varepsilon^2) & x \in \Omega, \end{cases} \tag{2.1}$$

where ε is a small parameter that can be arbitrarily close to *zero*, $k_{\text{in}} := \frac{\omega}{c_{\text{in}}}$ is the wavenumber inside Ω and γ is a bounded non negative function that vanishes outside B. The index of the medium q^ε can then be written in the form

$$q^\varepsilon = \begin{cases} q(x) & x \notin \Omega \\ \dfrac{\kappa_{\text{in}}^2}{\kappa^2}(1 + \dfrac{i}{\kappa_{\text{in}}^2 \varepsilon^2}) & x \in \Omega \end{cases}$$

where q is a bounded function with non negative imaginary part and equal to 1 outside B. We shall denote by q_r the real part of q^ε, which is a bounded function independent of ε.

As explained in Section 1.2.2, the scattering problem can be formulated inside a ball B_R of radius R (large enough to enclose the domain B) with the aid of the DtN map T_R (see (1.23) as the problem of seeking $u^\varepsilon \in H^1(B_R)$ such that

$$\begin{cases} \Delta u^\varepsilon + \kappa^2 q^\varepsilon u^\varepsilon = f & \text{in } B_R, \\ \partial_r u^\varepsilon - T_R(u^\varepsilon{}_{|S_R}) = g & \text{on } \partial B_R, \end{cases} \tag{2.2}$$

where $f \in L^2(B_R)$ and $g \in H^{-\frac{1}{2}}(\partial B_R)$ are source terms that may be linked with some incident field and are independent of ε. We shall further assume that the support of f is compactly embedded into $B_R \setminus \overline{\Omega}$.

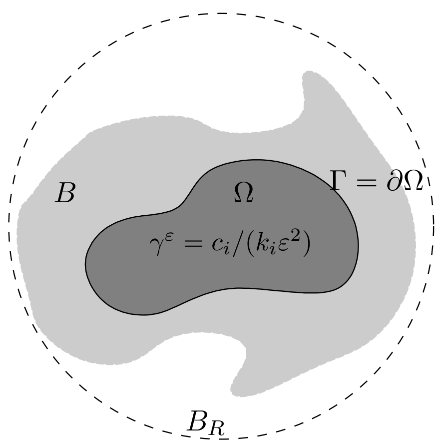


Figure 2.1: Geometry of the medium

Remark 2.1 *The parameter ε (that has the dimension of a length) is commonly denoted by the skin depth. It represents the width of the penetrable (boundary) region inside Ω.*

Roughly speaking, the goal of deriving *GIBCs* is to be able to approximate the restriction u_e^ε of u^ε to the exterior domain $B_R \setminus \Omega$ without having to solve the (singular perturbed) equations inside Ω. In order to do so, it is useful to rewrite problem (2.2) as a transmission problem between $u^\varepsilon_{\mathrm{in}} = u^\varepsilon_{|\Omega}$ and $u^\varepsilon_e = u^\varepsilon_{|B_R\setminus\Omega}$:

$$\left\{\begin{array}{lll} (i) & \Delta u^\varepsilon_e + \kappa^2 q\, u^\varepsilon_e = f, & \text{in } \ B_R \setminus \Omega, \\[2mm] (ii) & \Delta u^\varepsilon_{\mathrm{in}} + \kappa^2_{\mathrm{in}}\, u^\varepsilon_{\mathrm{in}} + \dfrac{i}{\varepsilon^2}\, u^\varepsilon_{\mathrm{in}} = 0, & \text{in } \ \Omega, \\[2mm] (iii) & \partial_r u^\varepsilon_e - T_R(u^\varepsilon_e) = g, & \text{on } \ \partial B_R, \\[2mm] (vi) & u^\varepsilon_{\mathrm{in}} = u^\varepsilon_e, & \text{on } \ \Gamma, \\[2mm] (v) & \partial_n u^\varepsilon_{\mathrm{in}} = \partial_n u^\varepsilon_e, & \text{on } \ \Gamma, \end{array}\right. \tag{2.3}$$

where n denotes here and all over this chapter a unitary normal field on Γ directed to the interior of Ω.

2.2.1 Existence-Uniqueness-Stability

This section is devoted to some basic theoretical results relative to problem (2.2). These results form the basis of our subsequent asymptotic analysis.

Theorem 2.1 *There exists a unique solution $u^\varepsilon \in H^1(B_R)$ to problem (2.2). Moreover, there exits a constant $C > 0$ independent of ε (and u^ε, f and g) such that*

$$\|u^\varepsilon\|_{L^2(B_R)} \leq C \left(\|f\|_{L^2(B_R)} + \|g\|_{H^{-\frac{1}{2}}(\partial B_R)} \right). \tag{2.4}$$

Proof. The existence and uniqueness proof has been already given in Section 1.2. According to Theorem 1.4, there exists a constant C^ε independent of u and f and g such that

$$\|u^\varepsilon\|_{L^2(B_R)} \leq C^\varepsilon \left(\|f\|_{L^2(B_R)} + \|g\|_{H^{-\frac{1}{2}}(\partial B_R)} \right).$$

We shall prove that we can choose the constant C^ε to be bounded with respect to ε. If the latter is not possible, then there exists f and g such that $\|u^\varepsilon\|_{L^2(B_R)} \to \infty$ as $\varepsilon \to 0$. Setting

$$\tilde{u}^\varepsilon := u^\varepsilon / \|u^\varepsilon\|_{L^2(B_R)}, \quad \tilde{f}^\varepsilon := f / \|u^\varepsilon\|_{L^2(B_R)}, \quad \text{and} \quad \tilde{g}^\varepsilon := g / \|u^\varepsilon\|_{L^2(B_R)},$$

one has in particular $\|\tilde{u}^\varepsilon\|_{L^2(B_R)} = 1$ while $\|\tilde{f}^\varepsilon\|_{L^2(B_R)} \to 0$ and $\|\tilde{g}^\varepsilon\|_{H^{-\frac{1}{2}}(\partial B_R)} \to 0$ as $\varepsilon \to 0$. Moreover, equation (2.2) yields

$$\left\{\begin{array}{ll} \Delta \tilde{u}^\varepsilon + \kappa^2 q^\varepsilon \tilde{u}^\varepsilon = \tilde{f}^\varepsilon & \text{in } B_R, \\[2mm] \partial_r \tilde{u}^\varepsilon - T_R(\tilde{u}^\varepsilon) = \tilde{g}^\varepsilon & \text{on } \partial B_R, \end{array}\right. \tag{2.5}$$

Consequently (after multiplying the first equation by $\overline{\tilde{u}^\varepsilon}$, integrate over B_R and use the Green formula)

$$-\int_{B_R} |\nabla \tilde{u}^\varepsilon|^2 \, dx + \kappa^2 \int_{B_R} q^\varepsilon \, |\tilde{u}^\varepsilon|^2 \, dx + \langle T_R(\tilde{u}^\varepsilon), u^\varepsilon \rangle_{\partial B_R} = \int_{B_R} \tilde{f}^\varepsilon \, \overline{\tilde{u}^\varepsilon} \, dx - \langle \tilde{g}^\varepsilon, \tilde{u}^\varepsilon \rangle_{\partial B_R} \tag{2.6}$$

Taking the real part of (2.6) yields

$$\int_{B_R} (|\nabla \tilde{u}^\varepsilon|^2 + |\tilde{u}^\varepsilon|^2)\, dx - \mathcal{R}e\, \langle T_R(\tilde{u}^\varepsilon), u^\varepsilon \rangle_{\partial B_R}$$
$$= \int_{B_R} (\kappa^2 q_r + 1)|\tilde{u}^\varepsilon|^2\, dx - \mathcal{R}e \left(\int_{B_R} \tilde{f}^\varepsilon\, \overline{\tilde{u}^\varepsilon}\, dx - \langle \tilde{g}^\varepsilon, \tilde{u}^\varepsilon \rangle_{\partial B_R} \right)$$

Since $\mathcal{R}e\, \langle T_R(\varphi), \varphi \rangle_{\partial B_R} \leq 0$, one easily deduces from the previous identity and trace theorems that $\tilde{u}^\varepsilon$ is a bounded sequence in $H^1(B_R)$. Hence one can assume that, up to the extraction of a subsequence, $\tilde{u}^\varepsilon \to \tilde{u}$ weakly in $H^1(B_R)$ and strongly in $L^2(B_R)$.
We first have that $\|\tilde{u}\|_{L^2(B_R)} = 1$. Taking the limit in (2.5), restricted to $B_R \setminus \Omega$, yields

$$\begin{cases} \Delta \tilde{u} - \kappa^2 q \tilde{u} = 0, & \text{in } \ B_R \setminus \Omega, \\ \partial_r \tilde{u} - T_R(\tilde{u}) = 0, & \text{on } \ \partial B_R. \end{cases} \tag{2.7}$$

On the other hand, taking the imaginary part in (2.6) shows in particular that

$$\frac{1}{\varepsilon^2} \|\tilde{u}^\varepsilon\|^2_{L^2(\Omega)} \leq \mathcal{I}m \left(\int_{B_R} \tilde{f}^\varepsilon\, \overline{\tilde{u}^\varepsilon}\, dx - \langle \tilde{g}^\varepsilon, \tilde{u}^\varepsilon \rangle_{\partial B_R} \right).$$

Thus $\tilde{u}^\varepsilon \to 0$ in $L^2(\Omega)$ as $\varepsilon \to 0$. Hence $\tilde{u} = 0$ in Ω and in particular

$$\tilde{u} = 0 \quad \text{on} \ \ \partial \Omega.$$

Combined with (2.7) and the uniqueness result of Theorem 1.3, this condition shows that $\tilde{u} = 0$ in $B_R \setminus \Omega$. We then obtain that $\tilde{u} = 0$ in B_R which contradicts $\|\tilde{u}\|_{L^2(B_R)} = 1$. □

Corollary 2.1 *there exits a constant $C > 0$ independent of ε (and u^ε, f and g) such that*

$$\left(\|u^\varepsilon\|_{L^2(B_R)} + \frac{1}{\varepsilon} \|u^\varepsilon\|_{L^2(\Omega)} \right) \leq C \left(\|f\|_{L^2(B_R)} + \|g\|_{H^{-\frac{1}{2}}(\partial B_R)} \right).$$

Proof. This corollary is a direct consequence of energy balance

$$- \int_{B_R} |\nabla u^\varepsilon|^2\, dx + \kappa^2 \int_{B_R} q^\varepsilon\, |u^\varepsilon|^2\, dx + \langle T_R(u^\varepsilon), u^\varepsilon \rangle_{\partial B_R} = \int_{B_R} f\, \overline{u^\varepsilon}\, dx - \langle \tilde{g}^\varepsilon, u^\varepsilon \rangle_{\partial B_R}$$

and the stability result of Theorem 2.1. □

Corollary 2.1 shows in particular that the solution converges to 0 like $O(\varepsilon)$ inside Ω, at least in the L^2 sense. This $O(\varepsilon)$ L^2-interior estimate is in fact not optimal. A sharper result, later given by Lemma 2.1, shows that $\|u^\varepsilon\|_{L^2(\Omega)}$ is $O(\varepsilon^{3/2})$ (see Remark 2.11).

2.2.2 Exponential convergence to 0 inside Ω

It is shown that the norm of the solution in a domain strictly interior to Ω goes to 0 faster than any power of ε. This is a first way to express how the interior solution concentrates near the boundary Γ. Let us indicate that this result will also be a consequence of the asymptotic analysis performed in next sections, however the methodology is more complex. The proof given here is a direct one and is independent of the subsequent analysis. The precise result is the following:

Theorem 2.2 *For any $\bar{\nu} > 0$ small enough so that $\Omega^{\bar{\nu}} := \{x \in \Omega; B(x, \bar{\nu}) \subset \Omega\}$ is a non-empty set, where $B(x, \bar{\nu})$ denotes the closed ball of center x and radius $\bar{\nu}$, there exist two positive constants $C_{\bar{\nu}}$ and $c_{\bar{\nu}}$ independent of ε such that*

$$\|u^\varepsilon\|_{H^1(\Omega^{\bar{\nu}})} \leq C_{\bar{\nu}} \exp(-c_{\bar{\nu}}/\varepsilon) \left(\|f\|_{L^2(B_R)} + \|g\|_{H^{-\frac{1}{2}}(\partial B_R)} \right).$$

Proof. One possible proof of this estimate can be obtained by using the integral representation of the solution inside Ω and the result of Corollary 2.1 . We shall present here an alternative variational approach has the advantage of also applying to the case of variable coefficients.

Let us introduce a cut-off function $\phi_{\bar{\nu}} \in C^\infty(B_R)$ such that

$$\phi_{\bar{\nu}}(x) = 0 \;\text{ in }\; B_R \setminus \Omega\,, \qquad \phi_{\bar{\nu}}(x) = \beta^{\bar{\nu}} \;\text{ in }\; \Omega^{\bar{\nu}},$$

where the constant $\beta^{\bar{\nu}} > 0$ is chosen such that

$$\|\nabla \phi_{\bar{\nu}}\|_\infty^2 < \frac{1}{4}\,. \tag{2.8}$$

We set $v^\varepsilon = \exp(\phi_{\bar{\nu}}(x)/\varepsilon) u^\varepsilon$. Straightforward calculations show that

$$\Delta u^\varepsilon = \exp(-\phi_{\bar{\nu}}(x)/\varepsilon) \left(\Delta v^\varepsilon - \frac{1}{\varepsilon}(2\nabla\phi_{\bar{\nu}} \cdot \nabla v^\varepsilon + \Delta\phi_{\bar{\nu}}\, v^\varepsilon) + \frac{|\nabla\phi_\delta|^2}{\varepsilon^2} v^\varepsilon \right).$$

Hence v^ε satisfies

$$\begin{cases} \Delta v^\varepsilon - \dfrac{2}{\varepsilon}\nabla\phi_{\bar{\nu}} \cdot \nabla v^\varepsilon + \left(\kappa^2 q^\varepsilon - \dfrac{\Delta\phi_{\bar{\nu}}}{\varepsilon} + \dfrac{|\nabla\phi_{\bar{\nu}}|^2}{\varepsilon^2} \right) v^\varepsilon = f & \text{in } B_R \\ \partial_n v^\varepsilon - T_R(v^\varepsilon) = g & \text{on } \partial B_R. \end{cases} \tag{2.9}$$

Multiplying the first equation in (2.9) by $\overline{v^\varepsilon}$, integrating by parts in B_R yields, using the fact that $v^\varepsilon = u^\varepsilon$ in $B_R \setminus \Omega$,

$$\begin{aligned} &\int_\Omega |\nabla v^\varepsilon|^2\, dx + \frac{2}{\varepsilon}\int_\Omega \nabla\phi_{\bar{\nu}} \cdot \nabla v^\varepsilon\, \bar{v}^\varepsilon\, dx + \int_\Omega \left(-\kappa_{\text{in}}^2 + \frac{\Delta\phi_{\bar{\nu}}}{\varepsilon} - \frac{|\nabla\phi_{\bar{\nu}}|^2 - i}{\varepsilon^2} \right) |v^\varepsilon|^2\, dx \\ &\quad = -\int_{B_R} f\, \bar{u^\varepsilon}\, dx + \int_{B_R\setminus\Omega} \left(\kappa^2 q |u_e^\varepsilon|^2 - |\nabla u^\varepsilon|^2 \right) dx + \langle T_R(u^\varepsilon) + g, u^\varepsilon \rangle_{\partial B_R} \end{aligned} \tag{2.10}$$

Let us denote by L_ε the right hand side of the previous equality. According to Corollary 2.1, there exists a constant C independent of ε such that

$$|L_\varepsilon| \leq C \left(\|f\|_{L^2(B_R)} + \|g\|_{H^{-\frac{1}{2}}(\partial B_R)} \right).$$

On the other hand, thanks to inequality (2.8), using $|b-a| \geq |b| - |a|$, we have the lower bound:

$$\left| \frac{|\nabla\phi_{\bar\nu}|^2 - i}{\varepsilon^2} - \left(\frac{\Delta\phi_{\bar\nu}}{\varepsilon} - \kappa_{\text{in}}^2 \right) \right| \geq \frac{1}{\varepsilon^2} - \frac{|\Delta\phi_{\bar\nu}|}{\varepsilon} - \kappa_{\text{in}}^2.$$

Therefore, taking the modulus of (2.10) yields

$$\begin{vmatrix} \|\nabla v^\varepsilon\|^2_{L^2(\Omega)} & - & \dfrac{2}{\varepsilon} \|\nabla\phi_{\bar\nu}\|_{L^\infty} \|\nabla v^\varepsilon\|_{L^2(\Omega)} \|v^\varepsilon\|_{L^2(\Omega)} \\ & + & \left(\dfrac{1}{\varepsilon^2} - \dfrac{\|\Delta\phi_{\bar\nu}\|_\infty}{\varepsilon} - \kappa_{\text{in}}^2 \right) \|v^\varepsilon\|^2_{L^2(\Omega)} \leq |L_\varepsilon|. \end{vmatrix}$$

Thanks to the inequality,

$$\frac{2}{\varepsilon} \|\nabla\phi_{\bar\nu}\|_{L^\infty} \|\nabla v^\varepsilon\|_{L^2(\Omega)} \|v^\varepsilon\|_{L^2(\Omega)} \leq \frac{1}{2} \|\nabla v^\varepsilon\|^2_{L^2(\Omega)} + \frac{2}{\varepsilon^2} \|\nabla\phi_{\bar\nu}\|^2_{L^\infty} \|v^\varepsilon\|^2_{L^2(\Omega)},$$

and inequality (2.8), one gets

$$\frac{1}{2} \|\nabla v^\varepsilon\|^2_{L^2(\Omega)} + \left(\frac{1}{2\varepsilon^2} - \frac{\|\Delta\phi_{\bar\nu}\|_\infty}{\varepsilon} - \kappa_{\text{in}}^2 \right) \|v^\varepsilon\|^2_{L^2(\Omega)} \leq |L_\varepsilon|. \tag{2.11}$$

Finally, for ε small enough so that $\dfrac{1}{2\varepsilon^2} - \dfrac{\|\Delta\phi_{\bar\nu}\|_\infty}{\varepsilon} - \kappa_{\text{in}}^2 \geq \dfrac{1}{4\varepsilon^2}$, inequality (2.11) yields

$$\frac{1}{2} \|\nabla v^\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{4\varepsilon^2} \|v^\varepsilon\|^2_{L^2(\Omega)} \leq |L_\varepsilon| \leq C \left(\|f\|_{L^2(B_R)} + \|g\|_{H^{-\frac{1}{2}}(\partial B_R)} \right).$$

This proves the Theorem with $c_{\bar\nu} = \beta^{\bar\nu}$ since

$$\begin{cases} \|u^\varepsilon\|_{L^2(\Omega^{\bar\nu})} \leq \exp\left(-\beta^{\bar\nu}/\varepsilon\right) \|v^\varepsilon\|_{L^2(\Omega)} \,, \\ \|\nabla u^\varepsilon\|_{L^2(\Omega^{\bar\nu})} \leq \exp\left(-\beta^{\bar\nu}/\varepsilon\right) \|\nabla v^\varepsilon\|_{L^2(\Omega)} . \end{cases}$$

□

2.2.3 A numerical illustration of the boundary layer effect

As an illustration of the boundary layer effect we here present the results some 2-D numerical simulations. More precisely, we consider the scattering of an incident plane wave propagating along the x_1 axis in the direction $x_1 > 0$ by an absorbing disk Ω of center 0 and radius 1 (this scattering problem corresponds to $f = 0$ and g given by (1.26) with the incident wave being a plane wave). The total field is then of the form

$$u^\varepsilon(x) = \exp i\kappa x_1 + u_s^\varepsilon(x),$$

where u_s^ε satisfies the Sömmerfeld radiation condition. We used a wavenumber $\kappa = \kappa_{\text{in}} = 4\pi$ which corresponds to a wavelength $\lambda = 0.5$. The numerical computation has been done using higher order finite element method with curved elements. The domain of computation is reduced to the disk of radius 2 thanks to the DtN map T_R (evaluated using an integral representation of the solution, as used in [21]).

Figures 2.2 to 2.5 represent the real part of the solution u. One clearly observe how the penetrable region inside the absorbing disk decreases as the absorption coefficient γ^ε increases. The skin effect is clearly illustrated in Figure 2.6 where the modulus of the total field is represented along line $x_2 = 0$, showing the exponential decay of the solution inside Ω.

2.3 Expressions of the GIBCs, stability and error estimates

Before going to technical details behind the derivation and analysis of GIBCs, this section intends to give an overview of obtained various approximations and a discussion of their stability with respect to the small parameter. These approximate problems are made of the standard Helmholtz equation in the exterior domain $B_R \setminus \Omega$ coupled with appropriate radiation condition on ∂B_R:

$$\begin{cases} \Delta u^{\varepsilon,k} + \kappa^2 q\, u^{\varepsilon,k} = f & \text{in } B_R \setminus \Omega, \\ \partial_r u^{\varepsilon,k} - T_R(u^{\varepsilon,k}) = g & \text{on } \partial B_R, \end{cases} \tag{2.12}$$

all together complemented with a $GIBC$ on the interior boundary Γ. The integer k used in the notation $u^{\varepsilon,k}$ of the approximate solution refers to the order of the $GIBC$. The precise mathematical meaning of this order will be later clarified with some error estimates (see Theorem 2.3). Roughly speaking, a $GIBC$ of order k is a boundary condition that will provide a (sharp) $O(\varepsilon^{k+1})$ error (in an appropriate sense) between $u^{\varepsilon,k}$ and the exact solution u_e^ε.

Let us mention that, for a given integer k, there are many different possible $GIBC$s of order k. These $GIBC$s have the form of a linear relationship between the Dirichlet and Neumann boundary values, $u^{\varepsilon,k}$ and $\partial_n u^{\varepsilon,k}$ on Γ involving local surface (differential) operators.

The asymptotic method used for deriving these approximate models naturally leads to

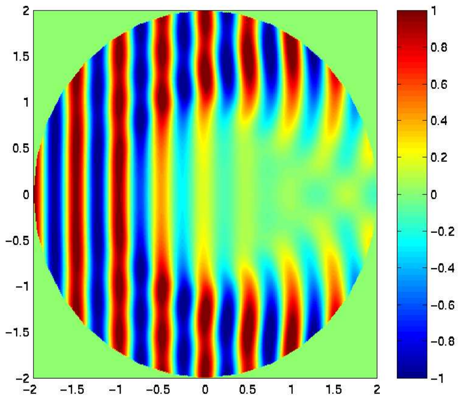


Figure 2.2: Total field for an absorbing disc: $\gamma^\varepsilon = 45$.

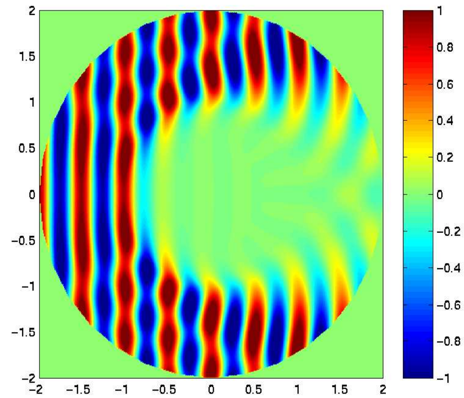


Figure 2.3: Total field for an absorbing disc: $\gamma^\varepsilon = 156$.

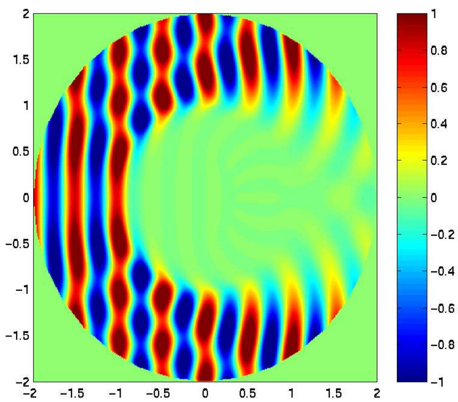


Figure 2.4: Total field for an absorbing disc: $\gamma^\varepsilon = 400$.

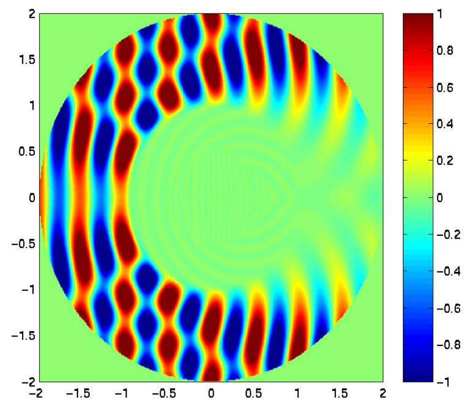


Figure 2.5: Total field for an absorbing disc: $\gamma^\varepsilon = 2500$.

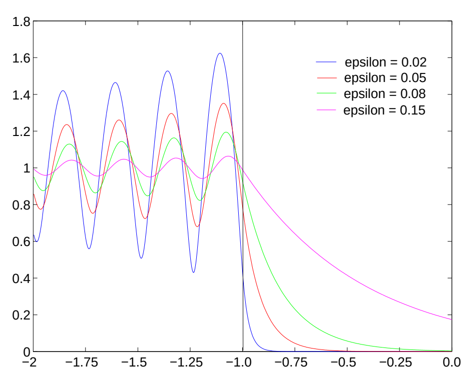


Figure 2.6: Plot of $x_1 \mapsto |u(x_1, 0)|$

Neumann-to-Dirichlet (NtD) $GIBC$s. These are the ones that we choose to present first in section 2.3.1. It will be clear in section 2.4 that we can derive, at least formally, a $GIBC$ of any order. However, the algebra becomes more and more involved as k increases, and it is perhaps impossible to write a general theory (existence, stability and error analysis) for any k. That is why we shall restrict ourselves to $GIBC$s of order $k = 0, 1, 2$ and 3.

In section 2.3.2 we shall indicate how one can easily obtain from (NtD) $GIBC$s some modified $GIBC$s that are of Dirichlet-to-Neumann (DtN) nature (as more commonly presented in the literature) or of mixed type.

It is not evident to determine which GIBC would be better (from practical point of view) for a given order. Several criteria has to be taken into account:

- the adequacy to a particular numerical method,
- the robustness of the $GIBC$,
- numerical accuracy.

It seems that only a detailed numerical study would fully answer this question. The second point will hereafter be slightly discussed from a theoretical point of view.

2.3.1 Neumann-to-Dirichlet GIBCs

Neumann-to-Dirichlet $GIBC$ can be seen as a (local) approximation of the exact Neumann-to-Dirichlet condition that can be written on Γ as

$$u_e^\varepsilon + \mathcal{D}^\varepsilon \partial_n u_e^\varepsilon = 0, \quad \text{on } \Gamma, \tag{2.13}$$

where $\mathcal{D}^\varepsilon : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$ is the boundary operator defined by:

$$\mathcal{D}^\varepsilon \varphi = u_{\text{in}}^\varepsilon(\varphi),$$

where $u_{\text{in}}^\varepsilon$ is the unique solution of the interior boundary value problem:

$$\begin{cases} \Delta u_{\text{in}}^\varepsilon(\varphi) + \kappa_{\text{in}}^2\, u_{\text{in}}^\varepsilon(\varphi) + \dfrac{i}{\varepsilon^2} u_{\text{in}}^\varepsilon(\varphi) = 0 & \text{in } \Omega, \\ \partial_n u_{\text{in}}^\varepsilon(\varphi) = -\varphi & \text{on } \Gamma. \end{cases} \tag{2.14}$$

The boundary condition (2.13) is nothing but a different formulation of the interface conditions (2.3-(vi)) and (2.3-(v)) and the obvious relation $u_{\text{in}}^\varepsilon + \mathcal{D}^\varepsilon \partial_n u_{\text{in}}^\varepsilon = 0$ on Γ.

The absorbing nature of the interior medium is equivalent to the following absorption property of the operator $\mathcal{D}^\varepsilon$ (this follows from Green's formula):

$$\forall\, \varphi \in H^{-\frac{1}{2}}(\Gamma), \quad \mathcal{I}m \, \langle\, \mathcal{D}^\varepsilon \varphi, \varphi \rangle_\Gamma \;=\; \frac{1}{\varepsilon^2} \int_\Omega |u_{\text{in}}^\varepsilon(\varphi)|^2 \, dx \geq 0. \tag{2.15}$$

It is well known that the operator $\mathcal{D}^\varepsilon$ is a non-local pseudo-differential operator whose explicit expression is not known (except in some special simplified cases: see Remark 2.3 below). Nevertheless, as $\varepsilon \to 0$, this operator becomes "almost local" (even differential), which is more or less intuitive according to the exponential interior decay of the solution with respect to ε^{-1}.

The expression of the Neumann-to-Dirichlet *GIBC* of order k that will obtained through the asymptotic analysis can be written in the form:

$$u^{\varepsilon,k} + \mathcal{D}^{\varepsilon,k}\partial_n u^{\varepsilon,k} = 0 \quad \text{on } \Gamma, \tag{2.16}$$

where, for $k = 0, 1, 2,$ and 3 the operator $\mathcal{D}^{\varepsilon,k}$ is given by:

$$\mathcal{D}^{\varepsilon,0} = 0, \tag{2.17}$$

$$\mathcal{D}^{\varepsilon,1} = \frac{\varepsilon}{\alpha}, \tag{2.18}$$

$$\mathcal{D}^{\varepsilon,2} = \frac{\varepsilon}{\alpha} - i\mathcal{H}\varepsilon^2, \tag{2.19}$$

$$\mathcal{D}^{\varepsilon,3} = \frac{\varepsilon}{\alpha} - i\mathcal{H}\varepsilon^2 - \frac{\alpha\,\varepsilon^3}{2}\left(3\mathcal{H}^2 - G + \kappa_{\text{in}}^2 + \Delta_\Gamma\right), \tag{2.20}$$

where, in expressions (2.17) to (2.20),

- $\alpha := \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$ denotes the complex square root of $-i$ with positive real part,
- $\mathcal{H}$ and G respectively denotes the mean and Gaussian curvatures of Γ,
- $\Delta_\Gamma := \operatorname{div}_\Gamma \nabla_\Gamma$ denotes the Laplace-Beltrami operator on Γ.

We refer to Chapter 3 for the definition of the above indicated geometrical quantities.

Remark 2.2 *It is worthwhile noticing that:*

- *One recovers of course the Dirichlet condition in the case $k = 0$ (the limit of $\mathcal{D}^{\varepsilon,k}$ when $\varepsilon \to 0$ is 0 for any k): the Dirichlet condition appears as the GIBC of order 0.*
- *The first three conditions are exactly of the same nature and correspond to a purely local impedance condition. The geometry of Γ only appears in the third condition through the mean curvature $\mathcal{H}$. Since the numerical approximation of the three conditions would therefore have the same cost (provided that $\mathcal{H}$ is easily computable) condition (2.19) would be preferred to (2.18).*
- *Condition (2.20) is more complicated. It involves a tangential differential operator along the boundary. This additional complexity will have of course consequences on the numerical approximation but also, as later shown, on the mathematical analysis.*

Remark 2.3 *The operators $\mathcal{D}^{\varepsilon,k}$ are of the form $\sum_{j\leq k} \varepsilon^j \ \mathcal{D}^j$ and thus appear as some truncated Taylor expansions of $\mathcal{D}^\varepsilon$. This is particularly clear in the (very special) case where Ω is the half-space $x_3 < 0$. In that case, the symbol of $\mathcal{D}^\varepsilon$ can be computed explicitly. More precisely, if one uses Fourier transform in the variables (x_2, x_2), one gets:*

$$\widehat{\mathcal{D}^\varepsilon}\varphi(\xi) = D^\varepsilon(\xi)\hat{\varphi}(\xi), \quad D^\varepsilon(\xi) := (|\xi|^2 - \kappa_{\rm in}^2 - i/\varepsilon^2)^{-\frac{1}{2}}, \quad \mathcal{I}m D^\varepsilon(\xi) > 0.$$

It is then straightforward to recover conditions (2.17) to (2.20) from successive Taylor expansions (for small ε) of $D^\varepsilon(\xi)$.

The main theoretical results of the asymptotic and stability analysis related to these conditions are summarized in the following theorem.

Theorem 2.3 *Let $k = 0, 1, 2$ or 3. Then, for sufficiently small ε, the boundary value problem ((2.12), (2.16)) has a unique solution $u^{\varepsilon,k} \in H^1(B_R \setminus \Omega)$. Moreover, there exists a constant C_k, independent of ε, such that*

$$\|u_e^\varepsilon - u^{\varepsilon,k}\|_{H^1(B_R\setminus\Omega)} \leq C_k \ \varepsilon^{k+1}. \tag{2.21}$$

Remark 2.4 *Let us mention that:*

- *For $k \leq 2$, the existence and uniqueness proof via Fredholm's alternative is trivial. The uniqueness proof relies on the following inequality (analogous to (2.15)):*

  $$\forall\ \varphi \in L^2(\Gamma), \quad \mathcal{I}m \int_\Gamma \mathcal{D}^{\varepsilon,k}\varphi \cdot \bar{\varphi}\ ds \geq -\beta \int_\Gamma |\varphi|^2\ ds, \quad (\textit{for some } \beta \geq 0), \tag{2.22}$$

  that expresses in particular the absorbing nature of the boundary condition and provides a sufficient (but not necessary) condition for the uniqueness of the solution. With this argument, it is easy to see that, for $k = 0, 1$, the existence and uniqueness result holds in fact for any $\varepsilon \geq 0$. For $k = 2$ one easily checks that (2.22) is true as soon as:

  $$\varepsilon\mathcal{H} \leq \frac{\sqrt{2}}{2}, \quad \textit{a. e. on } \Gamma. \tag{2.23}$$

  Notice that this inequality is algebraic. When Ω is convex, $\mathcal{H} \leq 0$ along Γ so that (2.23) induces no constraints on ε.

- *In the case $k = 3$, the proof is more complicated. In particular, there is no clear equivalent to inequality (2.22) and the uniqueness proof requires some more sophisticated argument (see lemma 2.4). This explains why in this case, one has no explicit upper bound for ε below which uniqueness is guaranteed.*

2.3.2 Modified GIBCs

Dirichlet to Neumann $GIBC$s. If we introduce $\mathcal{N}^\varepsilon := (\mathcal{D}^\varepsilon)^{-1}$, then the exact boundary condition for u_e^ε can be rewritten as :

$$\partial_n u_e^\varepsilon + \mathcal{N}^\varepsilon u_e^\varepsilon = 0, \quad \text{on } \Gamma, \tag{2.24}$$

In our terminology a DtN $GIBC$ will be of the form:

$$\partial_n u^{\varepsilon,k} + \mathcal{N}^{\varepsilon,k} u^{\varepsilon,k} = 0, \quad \text{on } \Gamma, \tag{2.25}$$

where $\mathcal{N}^{\varepsilon,k}$ denotes some local approximation of $\mathcal{N}^\varepsilon$. They can be directly obtained from $\mathcal{D}^{\varepsilon,k}$ by seeking local operators $\mathcal{N}^{\varepsilon,k}$ that formally satisfy:

$$\mathcal{D}^{\varepsilon,k} = \left(\mathcal{N}^{\varepsilon,k}\right)^{-1} + O(\varepsilon^{k+1}). \tag{2.26}$$

The expression of $\mathcal{N}^{\varepsilon,k}$ is derived from formal Taylor expansions of $(\mathcal{D}^{\varepsilon,k})^{-1}$. One gets,

$$\text{for } k=2, \qquad \mathcal{N}^{\varepsilon,2} = \frac{\alpha}{\varepsilon} + \mathcal{H}, \tag{2.27}$$

$$\text{for } k=3, \qquad \mathcal{N}^{\varepsilon,3} = \frac{\alpha}{\varepsilon} + \mathcal{H} - \frac{\varepsilon}{2\alpha}(\Delta_\Gamma + \mathcal{H}^2 - G + \kappa^2). \tag{2.28}$$

The important point here is that the results (existence, uniqueness and error estimates) stated in theorem 2.3 for problem ((2.12),(2.16)) still hold for problem ((2.12), (2.25)). We refer to section 2.6.

Remark 2.5 *Let us notice that:*

- *The difference between conditions (2.19) and (2.27) is quite small. In fact (2.27) can also be seen as a NtD $GIBC$ with $\mathcal{D}^{\varepsilon,2} = (\frac{\alpha}{\varepsilon} + \mathcal{H})^{-1}$! Notice however that for $k = 2$, the problem ((2.12), (2.25)) is well posed for any value of ε. This is a consequence of the following (uniform) absorption property*

$$\forall\, \varphi \in L^2(\Gamma), \quad \mathcal{I}m \int_\Gamma \varphi \cdot \overline{\mathcal{N}^{\varepsilon,2}\varphi}\, ds \geq \frac{\sqrt{2}}{2\varepsilon} \int_\Gamma |\varphi|^2\, ds. \tag{2.29}$$

- *The difference between conditions (2.20) and (2.28) is much more important. This has consequences on both mathematical and numerical analyses.*

Robust $GIBC$s. As mentioned earlier, an important property of the "exact" impedance condition is what we shall refer to as *absorption property*. It can be formally formulated for $\mathcal{D}^\varepsilon$ (resp. $\mathcal{N}^\varepsilon$) as:

$$\mathcal{I}m \int_\Gamma \mathcal{D}^\varepsilon \varphi \cdot \overline{\varphi}\, ds \ \geq 0, \quad \left(\text{resp.} \quad \mathcal{I}m \int_\Gamma \varphi \cdot \overline{\mathcal{N}^\varepsilon \varphi}\, ds \ \geq 0\right), \tag{2.30}$$

for all $\varphi \in C^\infty(\Gamma)$ and all $\varepsilon > 0$.

Definition 2.1 *We shall say that a NtD GIBC of the form (2.13) (resp. a DtN GIBC of the form (2.24)) is robust if the absorption property (2.30) still holds for all $\varepsilon > 0$, when $\mathcal{D}^\varepsilon$ is replaced by $\mathcal{D}^{\varepsilon,k}$ (resp. $\mathcal{N}^\varepsilon$ is replaced by $\mathcal{N}^{\varepsilon,k}$).*

In particular, establishing robustness implies the well-posedness of the approximate problem for any $\varepsilon > 0$. With this respect, the second order NtD *GIBC* (2.19) is not robust since (2.30) is guaranteed only if $\varepsilon\mathcal{H} \leq \frac{\sqrt{2}}{2}$, a. e. on Γ, which is a constraint for non convex Ω. However, the second order DtN *GIBC* (2.27) is robust (thus can be seen as a robust version of (2.19)). Concerning the third order conditions, neither the NtD *GIBC* (2.20) nor the DtN *GIBC* (2.28) is robust. Indeed, one has the identities:

$$\int_\Gamma \mathcal{D}^{\varepsilon,3}\varphi \cdot \overline{\varphi}\, ds = \frac{\alpha\,\varepsilon^3}{2}\int_\Gamma |\nabla_\Gamma \varphi|^2\, ds + \varepsilon\bar{\alpha}\int_\Gamma [\,1 + \frac{\varepsilon\mathcal{H}}{\alpha} + i\,\frac{\varepsilon^2}{2}\,(3\mathcal{H}^2 - G + \kappa^2)\,]\,|\varphi|^2\, ds,$$
$$\int_\Gamma \varphi \cdot \overline{\mathcal{N}^{\varepsilon,3}\varphi}\, ds = \frac{\alpha\,\varepsilon}{2}\int_\Gamma |\nabla_\Gamma \varphi|^2\, ds + \frac{\bar{\alpha}}{\varepsilon}\int_\Gamma [\,1 + \frac{\varepsilon\mathcal{H}}{\bar{\alpha}} - i\,\frac{\varepsilon^2}{2}\,(\mathcal{H}^2 - G + \kappa^2)\,]\,|\varphi|^2\, ds.$$

from which one easily computes that (recall that $\alpha = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$):

$$\begin{cases} \mathcal{I}m\displaystyle\int_\Gamma \mathcal{D}^{\varepsilon,3}\varphi \cdot \overline{\varphi}\, ds = -\frac{\sqrt{2}\,\varepsilon^3}{4}\int_\Gamma |\nabla_\Gamma \varphi|^2\, ds + \frac{\varepsilon\sqrt{2}}{2}\int_\Gamma \rho_1^\varepsilon\,|\varphi|^2\, ds, \\ \mathcal{I}m\displaystyle\int_\Gamma \varphi \cdot \overline{\mathcal{N}^{\varepsilon,3}\varphi}\, ds = -\frac{\sqrt{2}\,\varepsilon}{4}\int_\Gamma |\nabla_\Gamma \varphi|^2\, ds + \frac{\sqrt{2}}{2\varepsilon}\int_\Gamma \rho_2^\varepsilon\,|\varphi|^2\, ds. \end{cases}$$

where the functions ρ_j^ε converge (uniformly on Γ) to 1 when ε tend to 0 (and are thus positive for ε small enough). The problem then is that the integrals in $|\nabla_\Gamma \varphi|^2$ come with the wrong sign.

As we shall now explain, it is possible to construct robust *GIBC*s of order 3. The idea is to use some appropriate Padé approximation of the imaginary part of the boundary operators that formally gives the same order of approximation but restore absorption property. Consider for instance the NtD *GIBC* of order 3. Indeed

$$\mathcal{I}m\mathcal{D}^{\varepsilon,3} = \varepsilon\frac{\sqrt{2}}{2}\left(1 - \varepsilon\sqrt{2}\mathcal{H} + \frac{\varepsilon^2}{2}(3\mathcal{H}^2 - G + \kappa^2 + \Delta_\Gamma)\right).$$

One can therefore formally write

$$\mathcal{I}m\mathcal{D}^{\varepsilon,3} = \varepsilon\,\frac{\sqrt{2}}{2}\,\left(1 + \frac{\varepsilon^2}{2}(3\mathcal{H}^2 - G + \kappa^2)\right)\left(1 - \varepsilon\sqrt{2}\mathcal{H} + \frac{\varepsilon^2}{2}\Delta_\Gamma\right) + O(\varepsilon^4).$$

Note that, as $\mathcal{H}^2 - G = \frac{1}{4}(c_1 - c_2)^2$, where c_1 and c_2 are the two principal curvatures along Γ (see Chapter 3), we have

$$(1 + \frac{\varepsilon^2}{2}(3\mathcal{H}^2 - G + \kappa^2)) > 0.$$

It is then sufficient to seek a positive approximation of $(1-\varepsilon\sqrt{2}\mathcal{H}+\frac{\varepsilon^2}{2}\Delta_\Gamma)$ which can be obtained by considering the formal inverse, namely

$$1-\varepsilon\sqrt{2}\mathcal{H}+\frac{\varepsilon^2}{2}\Delta_\Gamma=\left\{1+\varepsilon\sqrt{2}\mathcal{H}+\frac{\varepsilon^2}{2}(4\mathcal{H}^2-\Delta_\Gamma)\right\}^{-1}+O(\varepsilon^3).$$

Therefore,

$$\mathcal{I}m\mathcal{D}^{\varepsilon,3}=\varepsilon\frac{\sqrt{2}}{2}(1+\frac{\varepsilon^2}{2}\ (3\mathcal{H}^2-G+\kappa^2))\ (1+\varepsilon\sqrt{2}\mathcal{H}+\frac{\varepsilon^2}{2}(4\mathcal{H}^2-\Delta_\Gamma))^{-1}+O(\varepsilon^4).$$

A robust NtD-like $GIBC$ of order 3 is obtained by replacing $\mathcal{D}^{\varepsilon,3}$ by

$$\begin{aligned}\mathcal{D}_r^{\varepsilon,3}\ &:=\ \varepsilon\frac{\sqrt{2}}{2}\left(1-\frac{\varepsilon^2}{2}(3\mathcal{H}^2-G+\kappa^2+\Delta_\Gamma)\right)\\ &\quad+i\varepsilon\frac{\sqrt{2}}{2}(1+\frac{\varepsilon^2}{2}\ (3\mathcal{H}^2-G+\kappa^2))\ (1+\varepsilon\sqrt{2}\mathcal{H}+\frac{\varepsilon^2}{2}(4\mathcal{H}^2-\Delta_\Gamma))^{-1}.\end{aligned}\tag{2.31}$$

This expression will be used in practice in the following sense:

$$\mathcal{D}_r^{\varepsilon,3}\varphi:=\varepsilon\frac{\sqrt{2}}{2}\left(1-\frac{\varepsilon^2}{2}(3\mathcal{H}^2-G+\kappa^2+\Delta_\Gamma)\right)\varphi+i\varepsilon\frac{\sqrt{2}}{2}(1+\frac{\varepsilon^2}{2}(3\mathcal{H}^2-G+\kappa^2))\psi\tag{2.32}$$

where ψ is solution to:

$$-\frac{\varepsilon^2}{2}\Delta_\Gamma\psi+(1+\varepsilon\sqrt{2}\mathcal{H}+2\varepsilon^2\mathcal{H}^2)\psi=\varphi.\tag{2.33}$$

One can easily verify that

$$\int_\Gamma \mathcal{D}_r^{\varepsilon,3}\varphi\cdot\overline{\varphi}\ ds=\varepsilon\frac{\sqrt{2}}{2}(1+\frac{\varepsilon^2}{2}(3\mathcal{H}^2-G+\kappa^2))\int_\Gamma(1+\varepsilon\sqrt{2}\mathcal{H}+\varepsilon^2 2\mathcal{H}^2)|\psi|^2+\frac{\varepsilon^2}{2}|\nabla_\Gamma\psi|^2\ ds.\tag{2.34}$$

The right hand side is non negative for all ε whence the absorption property for $\mathcal{D}_r^{\varepsilon,3}$.

Of course one can follow a similar procedure to derive robust DtN third order $GIBC$. The expression of this condition is based on the approximation:

$$\mathcal{I}m\mathcal{N}^{\varepsilon,3}=-\frac{\sqrt{2}}{2\varepsilon}(1+\frac{\varepsilon^2}{2}(\mathcal{H}^2-G+\kappa^2))\ \{1-\frac{\varepsilon^2}{2}\Delta_\Gamma\}^{-1}+O(\varepsilon^2).\tag{2.35}$$

Hence, replacing $\mathcal{N}^{\varepsilon,3}$ by

$$\begin{aligned}\mathcal{N}_r^{\varepsilon,3}\ &:=\ \frac{\sqrt{2}}{2\varepsilon}\left(1+\varepsilon\sqrt{2}\mathcal{H}-\frac{\varepsilon^2}{2}(\mathcal{H}^2-G+\kappa^2+\Delta_\Gamma)\right)\\ &\quad-i\frac{\sqrt{2}}{2\varepsilon}(1+\frac{\varepsilon^2}{2}(\mathcal{H}^2-G+\kappa^2))\ \{1-\frac{\varepsilon^2}{2}\Delta_\Gamma\}^{-1},\end{aligned}\tag{2.36}$$

in (2.28) gives another third order DtN *GIBC*. This condition is robust in view of the following identity, where the right hand side is non negative for all ε,

$$\mathcal{I}m \int_\Gamma \varphi \cdot \overline{\mathcal{N}^{\varepsilon,3}\varphi}\, ds = \frac{\sqrt{2}}{2\varepsilon} \int_\Gamma (1 + \frac{\varepsilon^2}{2}(\mathcal{H}^2 - G + \kappa^2)) \left(|\psi|^2 + \frac{\varepsilon^2}{2} |\nabla_\Gamma \psi|^2 \right) ds \qquad (2.37)$$

where ψ is solution to

$$-\frac{\varepsilon^2}{2}\Delta_\Gamma \psi + \psi = \varphi.$$

Remark 2.6 *As one can notice, there is no unique manner to derive GIBC, and even robust NtD (or DtN) GIBC. Notice also that the proposed third order ones involve a fourth order surface differential operator. It does not seem easy to derive a robust third order GIBC with only second order differential operator.*

2.4 Formal derivation of the GIBC

2.4.1 Preliminary material

Parametric coordinates. We refer to Chapter 3 for the details and notation related to this paragraph.
Let n be the inward normal field defined on $\partial\Omega$ and let $\bar{\nu}$ be a given positive constant chosen to be sufficiently small so that

$$\Omega^{\bar{\nu}} = \{x \in \Omega \; ; \; \mathrm{dist}(x, \Gamma) < \bar{\nu}\}$$

can be uniquely parameterized by the tangential coordinate x_Γ on Γ and the normal coordinate $\nu \in (0, \bar{\nu})$ through

$$x = x_\Gamma + \nu n(x_\Gamma), \quad x \in \Omega^{\bar{\nu}}. \qquad (2.38)$$

We recall that if v is a regular scalar function defined in $\Omega^{\bar{\nu}}$ and $\tilde{v}$ the function defined on $\Gamma \times (0,\, \nu_0)$ by

$$\tilde{v}(x_\Gamma, \nu) = v(x_\Gamma + \nu n),$$

then (see (3.8))

$$\begin{aligned} J_\nu^3\, \Delta v \;&=\; J_\nu\, \mathrm{div}_\Gamma\, (I_\Gamma + \nu\mathcal{M})^2 \nabla_\Gamma \tilde{v} - (\nabla_\Gamma J_\nu) \cdot (I_\Gamma + \nu\mathcal{M})^2 \nabla_\Gamma \tilde{v} \\ &\quad + 2J_\nu^2\, (\mathcal{H} + \nu G)\, \partial_\nu \tilde{v} + J_\nu^3\, \partial_{\nu\nu}^2 \tilde{v}. \end{aligned} \qquad (2.39)$$

The asymptotic ansatz. As it is quite usual, the derivation of the approximate boundary conditions will be based on an ansatz about the solution, that is to say an a priori particular form of the solution dependence with respect to ε.
To formulate this ansatz, it is useful to introduce a cut off function $\chi \in C^\infty(\Omega)$ such that $\chi = 1$ in $\Omega^{\bar{\nu}/2}$ and $\chi = 0$ in $\Omega \setminus \Omega^{\bar{\nu}}$. In our ansatz we are not interested in describing the

exact asymptotic development of $(1-\chi)u_{\text{in}}^{\varepsilon}$ since we already know that this part exponentially converges to 0 with ε (this is Theorem 2.2).
For the remaining part of the solution, we postulate the following (polynomial) expansions:

$$u_e^{\varepsilon}(x) = u_e^0(x) + \varepsilon u_e^1(x) + \varepsilon^2 u_e^2(x) + \cdots \quad \text{for } x \in B_R \setminus \Omega \tag{2.40}$$

where u_e^{ℓ}, $\ell = 0, 1, \cdots$ are functions defined on $B_R \setminus \Omega$ and

$$\chi(x) u_{\text{in}}^{\varepsilon}(x) = u_{\text{in}}^0(x_\Gamma, \nu/\varepsilon) + \varepsilon u_{\text{in}}^1(x_\Gamma, \nu/\varepsilon) + \varepsilon^2 u_{\text{in}}^2(x_\Gamma, \nu/\varepsilon) + \cdots \quad \text{for } x \in \Omega^{\bar{\nu}} \tag{2.41}$$

where x, x_Γ and ν are as in (2.38) and where $u_{\text{in}}^{\ell}(x_\Gamma, \eta) : \Gamma \times \mathbb{R}^+ \mapsto \mathbb{C}$ are functions such that

$$\lim_{\eta \to \infty} u_{\text{in}}^{\ell}(x_\Gamma, \eta) = 0 \quad \text{for a.e. } x_\Gamma \in \Gamma. \tag{2.42}$$

The latter condition is linked with the boundary layer effect. It will ensure that the u_{in}^{ℓ}'s are exponentially decreasing with respect to η.

Remark 2.7 *Notice that the expansion (2.41) makes sense since the local coordinates (x_Γ, ν) can be used inside the support of χ.*

In the next section, we shall identify the set of equations satisfied by (u_e^{ℓ}) and (u_{in}^{ℓ}) and the formal expansions (2.40) and (2.41) will be justified in section 2.5.

It is useful to introduce the notation

$$\widetilde{u}_{\text{in}}^{\varepsilon}(x_\Gamma, \eta) := u_{\text{in}}^0(x_\Gamma, \eta) + \varepsilon u_{\text{in}}^1(x_\Gamma, \eta) + \varepsilon^2 u_{\text{in}}^2(x_\Gamma, \eta) + \cdots \quad (x_\Gamma, \eta) \in \Gamma \times \mathbb{R}^+, \tag{2.43}$$

so that ansatz (2.41) has to be understood as

$$\chi(x) u_{\text{in}}^{\varepsilon}(x) = \widetilde{u}_{\text{in}}^{\varepsilon}(x_\Gamma, \nu/\varepsilon) + O(\varepsilon^{\infty}) \quad \text{for } x \in \Omega^{\eta}. \tag{2.44}$$

2.4.2 Asymptotic formal matching

This procedure consists into inserting the different ansatz into the problem equations, then formally equating the same powers or ε.

Let us first consider the exterior field u_e^{ε}. It is clear that each of the terms u_e^k in the expansion satisfies the (outgoing) Helmholtz equation in $B_R \setminus \Omega$ (simply substitute (2.40) into (2.3)(i)):

$$\begin{cases} -\Delta u_e^k - \kappa^2 q u_e^k = f^k & \text{in } B_R \setminus \Omega, \\ \partial_r u_e^k - T_R(u_e^k) = g^k & \text{on } \partial B_R, \end{cases} \tag{2.45}$$

where (f^k, g^k) are the terms of asymptotic expansion of (f, g). Since we assumed that the latter is independent of ε then

$$(f^0, g^0) = (f, g), \quad \text{and} \quad (f^k, g^k) = (0, 0) \text{ for } k > 0.$$

Concerning the interior field, from (2.3-ii), (2.44) and the substitution $\nu = \varepsilon\eta$ in (2.39), we obtain the following equation:

$$\left| \begin{array}{l} -\dfrac{1}{\varepsilon^2}\, J_{\varepsilon\eta}^3\, \partial_{\eta\eta}^2 \widetilde{u}_{\rm in}^\varepsilon - \dfrac{2}{\varepsilon} J_{\varepsilon\eta}^2\, (\mathcal{H} + \varepsilon\eta G)\, \partial_\eta \widetilde{u}_{\rm in}^\varepsilon \\ \quad - J_{\varepsilon\eta}\, \mathrm{div}_\Gamma\, (I_\Gamma + \varepsilon\eta\mathcal{M})^2\, \nabla_\Gamma\, \widetilde{u}_{\rm in}^\varepsilon + \nabla_\Gamma\, J_{\varepsilon\eta} \cdot (I_\Gamma + \varepsilon\eta\mathcal{M})^2\, \nabla_\Gamma\, \widetilde{u}_{\rm in}^\varepsilon \\ \qquad\qquad\qquad + J_{\varepsilon\eta}^3 (-\kappa_{\rm in}^2 - \dfrac{i}{\varepsilon^2})\, \widetilde{u}_{\rm in}^\varepsilon = 0 \end{array} \right. \tag{2.46}$$

that can be rearranged in the following form after multiplying by ε^2:

$$\left| \begin{array}{rcl} (-\partial_{\eta\eta}^2 - i)\, \tilde{u}_{\rm in}^\varepsilon & = & (1 - J_{\varepsilon\eta}^3)(-\partial_{\eta\eta}^2 - i)\, \tilde{u}_{\rm in}^\varepsilon + 2\varepsilon J_{\varepsilon\eta}^2 (\mathcal{H} + \varepsilon\eta G)\, \partial_\eta \tilde{u}_{\rm in}^\varepsilon + \varepsilon^2 \kappa^2 J_{\varepsilon\eta}^3 \tilde{u}_{\rm in}^\varepsilon \\ & + & \varepsilon^2 J_{\varepsilon\eta}\, \mathrm{div}_\Gamma\, (I_\Gamma + \varepsilon\eta\mathcal{M})^2 \nabla_\Gamma\, - \varepsilon^2\, \nabla_\Gamma\, J_{\varepsilon\eta} \cdot (I_\Gamma + \varepsilon\eta\mathcal{M})^2 \nabla_\Gamma\, \tilde{u}_{\rm in}^\varepsilon. \end{array} \right. \tag{2.47}$$

Considering that J_ν is a polynomial of degree 2 in ν, (2.47) can be rewritten as:

$$(-\partial_{\eta\eta}^2 - i)\, \tilde{u}_{\rm in}^\varepsilon = \sum_{\ell=1}^{8}\, \varepsilon^\ell \mathcal{A}_\ell\, \tilde{u}_{\rm in}^\varepsilon, \quad \text{on } \Gamma \times \mathbb{R}^+, \tag{2.48}$$

where $\mathcal{A}_\ell$ are some partial differential operators in (x_Γ, η) that are independent of ε. Formal identification gives, after rather lengthy than complicated calculations,

$$\mathcal{A}_1 = 2\mathcal{H}\,\partial_\eta - 6\eta\,\mathcal{H}\,(-\partial^2_{\eta\eta} - i) \tag{2.49}$$

$$\mathcal{A}_2 = \Delta_\Gamma + \kappa^2 + 2\eta\,(G + 4\mathcal{H}^2)\partial_\eta - 3\eta^2\,(G + 4\mathcal{H}^2)(-\partial^2_{\eta\eta} - i) \tag{2.50}$$

$$\begin{aligned} \mathcal{A}_3 &= 2\eta\,\left[\,\mathcal{H}\Delta_\Gamma + \mathrm{div}_\Gamma\,(\mathcal{M}\nabla_\Gamma\,) - \nabla_\Gamma\,\mathcal{H}\cdot\nabla_\Gamma\, + 3\kappa^2\mathcal{H}\,\right] \\ &+ 4\eta^2\,\mathcal{H}\,\left[\,(3G + 2\mathcal{H}^2)\,\partial_\eta\,\right] - 4\eta^3\,\mathcal{H}\,(3G + 2\mathcal{H}^2)\,(-\partial^2_{\eta\eta} - i) \end{aligned} \tag{2.51}$$

$$\begin{aligned} \mathcal{A}_4 &= \eta^2\,\left[\,G\,\Delta_\Gamma + 4\mathcal{H}\,\mathrm{div}_\Gamma\,(\mathcal{M}\nabla_\Gamma\,) + \mathrm{div}_\Gamma\,(\mathcal{M}^2\nabla_\Gamma\,)\,\right] \\ &- \eta^2\,\left[\,\nabla_\Gamma\,G\cdot\nabla_\Gamma\, + 4\nabla_\Gamma\,\mathcal{H}\cdot(\mathcal{M}\nabla_\Gamma\,) - 3\kappa^2(G + 4\mathcal{H}^2)\,\right] \\ &+ 4\eta^3\,G\,(G + 4\mathcal{H}^2)\,\partial_\eta - 3\eta^4\,G\,(G + 4\mathcal{H}^2)\,(-\partial^2_{\eta\eta} - i) \end{aligned} \tag{2.52}$$

$$\begin{aligned} \mathcal{A}_5 &= 2\eta^3\,\left[\,G\,\mathrm{div}_\Gamma\,(\mathcal{M}\nabla_\Gamma\,) + \mathcal{H}\,\mathrm{div}_\Gamma\,(\mathcal{M}^2\nabla_\Gamma\,)\,\right] \\ &- 2\eta^3\,\left[\nabla_\Gamma\,G\cdot(\mathcal{M}\nabla_\Gamma\,) + \nabla_\Gamma\,\mathcal{H}\cdot(\mathcal{M}^2\nabla_\Gamma\,) - 2\kappa^2\,\mathcal{H}\,(3G + 2\mathcal{H}^2)\,\right] \\ &+ 10\eta^4\,G^2\mathcal{H}\,\partial_\eta - 6\eta^5\,G^2\mathcal{H}\,(-\partial^2_{\eta\eta} - i) \end{aligned} \tag{2.53}$$

$$\begin{aligned} \mathcal{A}_6 &= \eta^4\,\left[\,G\,\mathrm{div}_\Gamma\,(\mathcal{M}^2\nabla_\Gamma\,) - \nabla_\Gamma\,G\cdot(\mathcal{M}^2\nabla_\Gamma\,) + 3\kappa^2 G(G + 4\mathcal{H}^2)\,\right] \\ &+ 2\eta^5\,G^3\partial_\eta - \eta^6\,G^3(-\partial^2_{\eta\eta} - i) \end{aligned} \tag{2.54}$$

$$\mathcal{A}_7 = 6\eta^5\,\kappa^2 G^2\mathcal{H} \tag{2.55}$$

$$\mathcal{A}_8 = \eta^6\,\kappa^2 G^3 \tag{2.56}$$

Therefore, by substituting (2.43) in equation (2.48) and equating the terms of same order in ε, we obtain an induction on k that allows us to recursively determine the u^k_{in}'s as functions of η. With the convention $u^k_{\mathrm{in}} \equiv 0$ for $k < 0$, one can write this induction in the form

$$(-\partial^2_{\eta\eta} - i)\;u^k_{\mathrm{in}} = \sum_{\ell=1}^{8} \mathcal{A}_\ell\;u^{k-\ell}_{\mathrm{in}}, \qquad \text{on } \Gamma\times\mathbb{R}^+, \tag{2.57}$$

for all $k \geq 0$. For any $k \geq 0$, one assumes that the fields u^l_{in} and u^l_e are known for $l < k$ so that (2.57) can be seen as an ordinary differential equation in η for $\eta \in [0, +\infty[$ whose unknown $\eta \mapsto u^k_{\mathrm{in}}(x_\Gamma, \eta)$ (the variable x_Γ plays the role of a parameter). Since this equation is of order 2, in addition to the condition at infinity (2.42), the solution of (2.57) with respect to η requires one initial condition at $\eta = 0$. This condition will be provided by one of the two interface conditions (2.3-vi) and (2.3-v).

We choose here to use the condition (2.3-v) which provides us a non homogeneous Neumann condition at $\eta = 0$ whose right hand side will be given by the exterior field u_e^{k-1}, namely (substitute (2.40)-(2.41) into (2.3-v) and identify the series after the change of variable $\nu = \varepsilon\eta$)

$$\partial_\eta u_{\text{in}}^k(x_\Gamma, 0) = \partial_n u_e^{k-1}|_\Gamma(x_\Gamma), \quad x_\Gamma \in \Gamma. \tag{2.58}$$

With such a choice, the other condition (2.3-vi) will serve as a non homogeneous Dirichlet boundary condition for the exterior field u_e^k, to complete (2.45):

$$u_e^k|_\Gamma(x_\Gamma) = u_{\text{in}}^k(x_\Gamma, 0), \quad x_\Gamma \in \Gamma. \tag{2.59}$$

Remark 2.8 *Choosing (2.58) as the boundary condition for (2.57) will naturally lead to NtD GIBCs. The alternative choice (2.59) would naturally lead to DtN GIBCs. Our choice seems to be more natural because, thanks to the shift of index in (2.58), the right hand side really appears as something known from previous steps. Condition (2.59) appears more as a coupling condition!*

2.4.3 Expression of interior field inside the boundary layer

We are interested in getting analytic expression for the "interior fields" u_{in}^k by solving the boundary problem (in the variable η) constituted by equations (2.57), (2.58) and (2.42). To simplify the notation, we shall set:

$$du_{\text{in}}^k(x_\Gamma) := \partial_\eta u_{\text{in}}^k(x_\Gamma, 0), \quad x_\Gamma \in \Gamma. \tag{2.60}$$

Using standard techniques for linear differential equations, it is easy to prove that the solution u_{in}^k is of the form:

$$u_{\text{in}}^k(x_\Gamma, \eta) = P_{x_\Gamma}^k(\eta)\, e^{-\alpha\,\eta} \tag{2.61}$$

for all $k \geq 0$, where $P_{x_\Gamma}^k$ is a polynomial with respect to η of degree k whose coefficients are proportional to $du_{\text{in}}^0, \cdots, du_{\text{in}}^{k-1}$ (recall that $\alpha = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$). More precisely, these polynomials satisfy an (affine) induction of order 8, of the form:

$$P_{x_\Gamma}^k(\eta) = -\frac{1}{\alpha}\, du_{\text{in}}^{k-1}(x_\Gamma) + \mathcal{L}_k\left(P_{x_\Gamma}^{k-1}(\eta), ..., P_{x_\Gamma}^{k-7}(\eta)\right)$$

where $\mathcal{L}_k$ is a linear form on $\mathbb{C}^7$ whose coefficients are linear in the $du_{\text{in}}^l(x_\Gamma)$'s. We shall not give here the expression of $\mathcal{L}_k$ for any k but restrict ourselves to the first four functions u_{in}^k

(this is sufficient for $GIBC$s up to order 3)

$$u^0_{\text{in}}(x_\Gamma,\eta) = 0 \tag{2.62}$$

$$u^1_{\text{in}}(x_\Gamma,\eta) = -\frac{1}{\alpha}\, du^0_{\text{in}}(x_\Gamma)\, e^{-\alpha\eta} \tag{2.63}$$

$$u^2_{\text{in}}(x_\Gamma,\eta) = \left\{\left(-\frac{1}{\alpha}\, du^1_{\text{in}}(x_\Gamma) + \frac{\mathcal{H}}{\alpha^2}\, du^0_{\text{in}}(x_\Gamma)\right) + \eta\, \frac{\mathcal{H}}{\alpha}\, du^0_{\text{in}}(x_\Gamma)\right\} e^{-\alpha\eta} \tag{2.64}$$

$$\begin{aligned} u^3_{\text{in}}(x_\Gamma,\eta) = {} & \Big\{- \frac{1}{\alpha}\, du^2_{\text{in}}(x_\Gamma) + \frac{\mathcal{H}}{\alpha^2}\, du^1_{\text{in}}(x_\Gamma) \\ & - \frac{1}{2\alpha^3}\,(3\mathcal{H}^2 - G + \kappa^2)\, du^0_{\text{in}}(x_\Gamma) - \frac{1}{2\alpha^3}\, \Delta_\Gamma[du^0_{\text{in}}](x_\Gamma) \\ & + \eta\, [\, \frac{\mathcal{H}}{\alpha}\, du^1_{\text{in}}(x_\Gamma) - \frac{1}{2\alpha^2}\, (\Delta_\Gamma - G + 3\mathcal{H}^2 + \kappa^2)\, du^0_{\text{in}}(x_\Gamma)\,] \\ & + \eta^2\, \frac{1}{2\alpha}\, (G - 3\mathcal{H}^2)\, du^0_{\text{in}}(x_\Gamma)\, \Big\}\; e^{-\alpha\eta} \end{aligned} \tag{2.65}$$

2.4.4 Construction of the GIBCs

Let us first inductively check that, starting from $u^0_{\text{in}} = 0$ and u^0_e solution of the exterior Dirichlet problem the fields u^k_e and u^k_{in} are well defined.
Assume that u^ℓ_e and u^ℓ_{in} are known for $\ell \le k-1$.
The du^ℓ_{in}'s are known by (2.60), u^k_{in} is determined by the explicit expression (2.61) (and more precisely (2.62) to (2.65) for $k = 0, 1, 2, 3$).
Then, $u^k_e \in H^1(B_R \backslash \Omega)$ is determined as the unique solution of the boundary value problem

$$\begin{cases} \Delta u^k_e + \kappa^2 q\, u^k_e = f^k, & \text{in } \; B_R \setminus \Omega, \\ \partial_r u^k_e - T_R(u^k_e) = g^k, & \text{on } \; \partial B_R, \\ u^k_e = u^k_{\text{in}|\eta=0}\,, & \text{on } \; \Gamma. \end{cases} \tag{2.66}$$

Remark 2.9 *Since f is compactly supported in $B_R \setminus \Omega$, we (inductively) deduce from standard elliptic regularity that u^k_e is a smooth function in a neighborhood of Γ and also that $x_\Gamma \mapsto u^k_e(x_\Gamma, 0)$ is a smooth function on Γ.*

The $GIBC$ of order k is obtained by considering the truncated expansion:

$$\widetilde{u}^{\varepsilon,k} := \sum_{\ell=0}^{k} \varepsilon^\ell u^\ell_e \tag{2.67}$$

as an approximation of order k of u^ε_e.

For example, for $k = 0$, we have $\widetilde{u}^{\varepsilon,k} = u_e^0$ and from equations (2.59) and (2.62), we deduce that $\widetilde{u}^{\varepsilon,k} = 0$ on Γ. In this case we set $u^{\varepsilon,k} = \widetilde{u}^{\varepsilon,k}$ and, as emphasized in Remark 2.2, the Dirichlet condition:

$$u^{\varepsilon,k} = 0, \quad \text{on } \Gamma, \tag{2.68}$$

is a the $GIBC$ of order 0.

For larger k, another approximation is needed. The procedure main steps are the following. Using the second interface condition, namely (2.3-iv), one has

$$\widetilde{u}^{\varepsilon,k}|_\Gamma(x_\Gamma) = \sum_{\ell=0}^{k} \varepsilon^\ell u_{\text{in}}^\ell(x_\Gamma, 0) \quad \text{for} \;\; x_\Gamma \in \Gamma. \tag{2.69}$$

Substituting expressions (2.62)-(2.65) into (2.69) leads to a boundary condition of the form

$$\widetilde{u}^{\varepsilon,k} + \mathcal{D}^{\varepsilon,k}\, \partial_n \widetilde{u}^{\varepsilon,k} = \varepsilon^{k+1}\, g_k^\varepsilon \quad \text{on } \Gamma, \tag{2.70}$$

where $\mathcal{D}^{\varepsilon,k}$ is some boundary operator and where $\|g_k^\varepsilon\|_{H^m(\Gamma)}$ is uniformly bounded with respect to ε for any real m and for ε small enough.
The GIBC of order k that defines $u^{\varepsilon,k}$ (which is in general different from $\widetilde{u}^{\varepsilon,k}$) is then obtained by neglecting the right hand side of (2.70).

Obtaining (2.70) is the pure algebraic part of the work and we shall not give the details of the computations which are straightforward and could be automatized. Notice however that their complexity increases rapidly with k.

Exercise 2.1 *Prove that for $k \leq 3$, the operators $\mathcal{D}^{\varepsilon,k}$ are the ones announced in section 2.3.1 and that the reminders g_k^ε are given by:*

$$\left\{ \begin{array}{rcl} g_1^\varepsilon &=& \dfrac{1}{\alpha}\, \partial_n u_e^1, \\[2mm] g_2^\varepsilon &=& \dfrac{1}{\alpha}\, \partial_n u_e^2 - i\, \mathcal{H}\, \partial_n(u_e^1 + \varepsilon u_e^2), \\[2mm] g_3^\varepsilon &=& \dfrac{1}{\alpha}\, \partial_n u_e^3 - i\, \mathcal{H} \partial_n(u_e^2 + \varepsilon u_e^3) \\[2mm] && -\; \dfrac{1}{2}\, [\, \Delta_\Gamma \partial_n + (3\mathcal{H}^2 - G + \kappa^2)\partial_n\,]\, \left(u_e^1 + \varepsilon u_e^2 + \varepsilon^2 u_e^3\right). \end{array} \right. \tag{2.71}$$

2.5 Error analysis of NtD GIBCs

Our goal in this section is to estimate the difference

$$u_e^\varepsilon - u^{\varepsilon,k} \tag{2.72}$$

where $u^{\varepsilon,k}$ is the solution of the approximate problem ((2.12), (2.16)), whose well-posedness will be shown in section 2.5.2 (Lemma 2.4).
It is not evident how one can derive error estimates by working directly with the difference $u_e^\varepsilon - u^{\varepsilon,k}$. The route we shall follow make use of the truncated series $\widetilde{u}^{\varepsilon,k}$ introduced in Section 2.4.4 as an intermediate quantity. Therefore, the error analysis is split into two steps:

1. Estimate the difference $u_e^\varepsilon - \widetilde{u}^{\varepsilon,k}$; this is the object of Section 2.5.1, and more precisely of Lemma 2.1 and Corollary 2.2.
2. Estimate the difference $\widetilde{u}^{\varepsilon,k} - u^{\varepsilon,k}$; this is the object of Section 2.5.2 and more precisely of Lemma 2.6.

Estimates of theorem 2.3 are then a direct consequence of corollary 2.2 and Lemma 2.6.

Remark 2.10 *Notice that step 1 of the proof is completely independent on the GIBC and will be valid for any integer k. Also, for $k = 0$, the second step is useless since $\widetilde{u}^{\varepsilon,k} = u^{\varepsilon,k}$.*

2.5.1 Error analysis of the truncated expansions

Let us introduce the function $\widetilde{u}_\chi^{\varepsilon,k}(x) : B_R \mapsto \mathbb{C}$ such that

$$\widetilde{u}_\chi^{\varepsilon,k}(x) = \begin{cases} \displaystyle\sum_{\ell=0}^{k} \varepsilon^\ell u_e^\ell(x), & \text{for } x \in B_R \setminus \Omega, \\ \displaystyle\chi(x) \sum_{\ell=0}^{k} \varepsilon^\ell u_{\text{in}}^\ell(x_\Gamma, \nu/\varepsilon) & \text{for } x \in \Omega, \end{cases} \tag{2.73}$$

where χ, x_Γ and ν are as in section 2.4.1. The main result of this section is:

Lemma 2.1 *For any integer k, there exists a constant C_k independent of ε such that*

$$\begin{aligned} \|u^\varepsilon - \widetilde{u}_\chi^{\varepsilon,k}\|_{H^1(B_R)} &\leq C_k\, \varepsilon^{k+\frac{1}{2}}, \\ \|u^\varepsilon - \widetilde{u}_\chi^{\varepsilon,k}\|_{L^2(\Omega)} &\leq C_k\, \varepsilon^{k+\frac{3}{2}}, \\ \|u^\varepsilon - \widetilde{u}_\chi^{\varepsilon,k}\|_{L^2(\Gamma)} &\leq C_k\, \varepsilon^{k+1}. \end{aligned} \tag{2.74}$$

As an immediate corollary of this Lemma we obtain an $O(\varepsilon^{k+1})$ $H^1(B_R \setminus \Omega)$-error estimate for the "exterior field".

Corollary 2.2 *For any integer k, there exists a constant $\widetilde{C}_k$ independent of ε such that:*

$$\|u^\varepsilon - \widetilde{u}^{\varepsilon,k}\|_{H^1(B_R\setminus\Omega)} \leq \widetilde{C}_k\, \varepsilon^{k+1}. \tag{2.75}$$

Proof. Simply write

$$u^\varepsilon - \widetilde{u}^{\varepsilon,k} = u^\varepsilon - \widetilde{u}^{\varepsilon,k+1} + \varepsilon^{k+1} u_e^{k+1}$$

which yields, since $u^{\varepsilon,k+1} = u_\chi^{\varepsilon,k+1}$ in $B_R \setminus \Omega$,

$$\|u^\varepsilon - \widetilde{u}^{\varepsilon,k}\|_{H^1(B_R\setminus\Omega)} \le \|u^\varepsilon - \widetilde{u}_\chi^{\varepsilon,k+1}\|_{H^1(B_R\setminus\Omega)} + \varepsilon^{k+1}\, \|u_e^{k+1}\|_{H^1(B_R\setminus\Omega)},$$

that is to say, thanks to the first estimate of Lemma 2.1:

$$\|u^\varepsilon - \widetilde{u}^{\varepsilon,k}\|_{H^1(B_R\setminus\Omega)} \le C_k\, \varepsilon^{k+\frac{3}{2}} + \ \varepsilon^{k+1}\, \|u_e^{k+1}\|_{H^1(B_R\setminus\Omega)} \le \widetilde{C}_k\, \varepsilon^{k+1}.$$

□

Remark 2.11 *For $k=0$, since $\widetilde{u}_\chi^{\varepsilon,0} = 0$ inside Ω (cf (2.63)), one deduces from the second estimate of (2.74) that: $\|u^\varepsilon\|_{L^2(B_R\setminus\Omega)} \le C\, \varepsilon^{\frac{3}{2}}$.*

We shall prove first a technical trace lemma (Lemma 2.2) and a fundamental stability estimate (Lemma 2.3) that constitute the basic ingredients to the proof of Lemma 2.1.

Lemma 2.2 *Let O be a bounded open set of $\mathbb{R}^N$ with C^1 boundary, then there exists a constant C depending only on O such that*

$$\|u\|^2_{L^2(\partial O)} \le C\left(\|\nabla u\|_{L^2(O)}\, \|u\|_{L^2(O)} + \|u\|^2_{L^2(O)}\right), \quad \textit{for all } u \in H^1(O). \tag{2.76}$$

Proof. Assume first that $O = \mathbb{R}^N_+ := \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}+\}$ and let u in $C^\infty(\mathbb{R}^N_+)$ with compact support. Obviously

$$|u(x',0)|^2 = -2\int_0^\infty u\, \frac{\partial u}{\partial x_N}\, dx_N.$$

Therefore, using the Schwarz inequality,

$$\|u(\cdot,0)\|^2_{L^2(\mathbb{R}^{N-1})} \le 2\|\nabla u\|_{L^2(\mathbb{R}^N_+)}\, \|u\|_{L^2(\mathbb{R}^N_+)}. \tag{2.77}$$

Using the denseness of $C^\infty(\mathbb{R}^N_+)$ functions with compact support into $H^1(\mathbb{R}^N_+)$, we deduce that the previous inequality holds for all $u \in H^1(\mathbb{R}^N_+)$. Now let O be bounded open set of $\mathbb{R}^n$ and let χ a $C^\infty(B_R)$ cut off function such that

$$\chi(x) = 1 \quad \text{if } \operatorname{dist}(x, \partial O) < \bar{\nu}/2 \text{ and } \chi(x) = 0 \quad \text{if } \operatorname{dist}(x, \partial O) > \bar{\nu}$$

for a sufficiently small $\bar{\nu} > 0$. Using local parametric representations of suppχ, we deduce from (2.77) the existence of a constant C depending on ∂O and $\bar{\nu}$ such that

$$\|u\|^2_{L^2(\partial O)} \le C\, \|\nabla(\chi u)\|_{L^2(O)} \|\chi u\|_{L^2(O)},$$

whence the result of the Lemma with a different constant C. □

Lemma 2.3 *Let $v^\varepsilon \in H^1(B_R)$ satisfying*

$$\begin{cases} \Delta v^\varepsilon + \kappa^2 q\, v^\varepsilon = 0, & \text{in } \; B_R \setminus \Omega, \\ \partial_r v^\varepsilon - T_R(v^\varepsilon) = 0, & \text{on } \; \partial B_R, \end{cases} \tag{2.78}$$

and verifying the a priori estimate

$$\begin{aligned} \left| \int_{B_R} \left(|\nabla v^\varepsilon|^2 - \kappa^2 q_r |v^\varepsilon|^2 \right) dx \right. & - \left. \frac{i}{\varepsilon^2} \int_\Omega |v^\varepsilon|^2 \, dx \right| \\ & \leq A \left(\varepsilon^{s+\frac{1}{2}} \|v^\varepsilon\|_{L^2(\Gamma)} + \varepsilon^s \, \|v^\varepsilon\|_{L^2(\Omega)} \right), \end{aligned} \tag{2.79}$$

for some non negative constants A and s independent of ε. Then there exists a constant C independent of ε such that

$$\|v^\varepsilon\|_{H^1(B_R)} \leq C\, \varepsilon^{s+1}, \quad \|v^\varepsilon\|_{L^2(\Omega)} \leq C\, \varepsilon^{s+2}, \quad \|v^\varepsilon\|_{L^2(\Gamma)} \leq C\, \varepsilon^{s+\frac{3}{2}}, \tag{2.80}$$

for sufficiently small ε.

Proof. We first prove by contradiction that $\|v^\varepsilon\|_{L^2(B_R)} \leq C\, \varepsilon^{s+1}$. This is the main step of the proof. Let $w^\varepsilon = v^\varepsilon / \|v^\varepsilon\|_{L^2(B_R)}$ and assume that $\lambda^\varepsilon := \varepsilon^{-s-1} \|v^\varepsilon\|_{L^2(B_R)}$ is unbounded as $\varepsilon \to 0$. Estimate (2.79) (notice it is not homogeneous in v^ε) yields

$$\begin{aligned} \left| \int_{B_R} \left(|\nabla w^\varepsilon|^2 - \kappa^2 q_r |w^\varepsilon|^2 \right) dx \right. & - \left. \frac{i}{\varepsilon^2} \int_\Omega |w^\varepsilon|^2 \, dx \right| \\ & \leq \frac{A}{\lambda^\varepsilon} \left(\varepsilon^{-\frac{1}{2}} \|w^\varepsilon\|_{L^2(\Gamma)} + \varepsilon^{-1} \, \|w^\varepsilon\|_{L^2(\Omega)} \right), \end{aligned} \tag{2.81}$$

For sake of conciseness, we will denote by C a positive constant whose value may change from one line to another but remains independent of ε. For instance, (2.81) yields in particular, since $1/\lambda^\varepsilon$ is bounded,

$$\|w^\varepsilon\|^2_{L^2(\Omega)} \leq C\, \varepsilon^{\frac{3}{2}} \|w^\varepsilon\|_{L^2(\Gamma)} + C\, \varepsilon \|w^\varepsilon\|_{L^2(\Omega)}.$$

Next, we use Lemma 2.2 with $\mathcal{O} = \Omega$ to get

$$\|w^\varepsilon\|^2_{L^2(\Omega)} \leq C\, \varepsilon^{\frac{3}{2}} \|w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega)} \left(\|w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega)} + \|\nabla w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega)} \right) + C\, \varepsilon \|w^\varepsilon\|_{L^2(\Omega)},$$

which yields, after division by $\|w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega)}$,

$$\|w^\varepsilon\|^{\frac{3}{2}}_{L^2(\Omega)} \leq C_1\, \varepsilon \|w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega)} + C_2\, \varepsilon^{\frac{3}{2}} \|\nabla w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega)}. \tag{2.82}$$

Using Young's inequality $ab \leq 2/3 \; a^{3/2} + 1/3 \; b^3$ with $a = K^{-1}\varepsilon$ and $b = K \; \|w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega)}$ (where K is a positive constant to be fixed later) we can write

$$\varepsilon \|w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega)} \leq \frac{2}{3}\, K^{-\frac{3}{2}}\, \varepsilon^{\frac{3}{2}} + \frac{K^3}{3}\, \|w^\varepsilon\|^{\frac{3}{2}}_{L^2(\Omega)}. \tag{2.83}$$

Choosing $C_1 K^3 = 3/2$ and substituting (2.82) into (2.83), we deduce a first main inequality,

$$\|w^\varepsilon\|^{\frac{3}{2}}_{L^2(\Omega)} \leq C\ \varepsilon^{\frac{3}{2}}\ \left(1 + \|\nabla w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega)}\right). \tag{2.84}$$

Now, observe that another consequence of (2.81) is, since $\|w^\varepsilon\|_{L^2(B_R)} = 1$,

$$\|\nabla w^\varepsilon\|^2_{L^2(B_R)} \leq C \left(1 + \varepsilon^{-\frac{1}{2}}\|w^\varepsilon\|_{L^2(\Gamma)} + \varepsilon^{-1}\|w^\varepsilon\|_{L^2(\Omega)}\right). \tag{2.85}$$

On the other hand, using Lemma 2.2 once again, we have

$$\varepsilon^{-\frac{1}{2}}\|w^\varepsilon\|_{L^2(\Gamma)} \leq C\ \varepsilon^{\frac{1}{2}} \left\{\varepsilon^{-1}\|w^\varepsilon\|_{L^2(\Omega)}\right\} + C\ \left\{\varepsilon^{-1}\|w^\varepsilon\|_{L^2(\Omega)}\right\}^{\frac{1}{2}}\ \|\nabla w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega)},$$

which, for ε bounded, implies, using (2.85),

$$\|\nabla w^\varepsilon\|^2_{L^2(B_R)} \leq C + C\ \left\{\varepsilon^{-1}\|w^\varepsilon\|_{L^2(\Omega)}\right\} \left(1 + \|\nabla w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega)}\right). \tag{2.86}$$

Coming back to (2.84), we deduce that

$$\varepsilon^{-1}\|w^\varepsilon\|_{L^2(\Omega)} \leq C\ \left(1 + \|\nabla w^\varepsilon\|^{\frac{1}{3}}_{L^2(\Omega)}\right), \tag{2.87}$$

that we use in (2.86) to obtain

$$\|\nabla w^\varepsilon\|^2_{L^2(B_R)} \leq C \left(1 + \|\nabla w^\varepsilon\|^{\frac{2}{3}}_{L^2(\Omega)}\right).$$

This implies in particular that $\|\nabla w^\varepsilon\|_{L^2(B_R)}$ is uniformly bounded with respect to ε and therefore w^ε is a bounded sequence of $H^1(B_R)$. Up to an extracted subsequence, one can therefore assume that w^ε converges weakly in $H^1(B_R)$ and strongly $L^2(B_R)$ to some w with $\|w\|_{L^2(B_R)} = 1$.

From (2.84), we deduce that $w = 0$ in Ω. On the other hand, taking the weak limit in the equations satisfied by w^ε in $B_R \setminus \Omega$ and on ∂B_R, then using that $w \in H^1(B_R)$ one gets

$$\begin{cases} \Delta w + \kappa^2 q\, w = 0, & \text{in } \ B_R \setminus \Omega \\ \partial_r w - T_R(w) = 0 & \text{on } \ \partial B_R, \\ w = 0 & \text{on } \ \Gamma. \end{cases} \tag{2.88}$$

Therefore $w = 0$ in $B_R \setminus \Omega$ (by using Lemma 1.3). Hence $w = 0$ in B_R which contradicts $\|w\|_{L^2(B_R)} = 1$. Consequently

$$\|v^\varepsilon\|_{L^2(B_R)} \leq C\ \varepsilon^{s+1}. \tag{2.89}$$

Estimate (2.79) and Lemma 2.2 yields

$$\|v^\varepsilon\|^2_{L^2(\Omega)} \leq C \left(\varepsilon^{s+\frac{5}{2}}\|\nabla v^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega)}\|v^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega)} + \varepsilon^{s+2}\|v^\varepsilon\|_{L^2(\Omega)}\right), \tag{2.90}$$

and, using (2.89)

$$\|\nabla v^\varepsilon\|^2_{L^2(B_R)} \leq C\left(\varepsilon^{2s+2} + \varepsilon^{s+\frac{1}{2}}\|\nabla v^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega)}\|v^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega)} + \varepsilon^s\|v^\varepsilon\|_{L^2(\Omega)}\right). \tag{2.91}$$

Therefore, combining these two estimates, it is not difficult to obtain

$$\|v^\varepsilon\|^2_{L^2(\Omega)} + \varepsilon^2\|\nabla v^\varepsilon\|^2_{L^2(B_R)} \leq C\left(\varepsilon^{2s+4} + \varepsilon^{s+2}\left(\|v^\varepsilon\|_{L^2(\Omega)} + \varepsilon\|\nabla v^\varepsilon\|_{L^2(B_R)}\right)\right),$$

which yields

$$\|v^\varepsilon\|_{L^2(\Omega)} + \varepsilon\|\nabla v^\varepsilon\|_{L^2(B_R)} \leq C\varepsilon^{s+2}.$$

This corresponds to the first two estimates of (2.80). The third one is a direct consequence of these two estimates by the application of Lemma 2.2 to Ω. □

Remark 2.12 *Notice that since we simply used in the first step of the proof the fact that $1/\lambda^\varepsilon$ is bounded, we have proved in fact that*

$$\lim_{\varepsilon\to 0} \varepsilon^{-(s+1)}\,\|v^\varepsilon\|_{L^2(B_R)} = 0.$$

Proof of Lemma 2.1. Let us set $e^\varepsilon_k = u^\varepsilon - \widetilde{u}^{\varepsilon,k}_\chi$. The idea of the proof is to show that e^ε_k satisfies an a priori estimate of the type (2.79) and then to use the stability Lemma 2.3. To prove such an estimate, we shall use the equations satisfied by e^ε_k, respectively in Ω and $B_R \setminus \Omega$ as well as transmission conditions across Γ.

The exterior equation. By construction, $\widetilde{u}^{\varepsilon,k}_\chi$ satisfies in $B_R \setminus \Omega$ the non homogeneous Helmholtz equation with the radiation boundary condition on ∂B_R and right hand sides f and g (this is a direct consequence of (2.45) for each k). Hence, $e^\varepsilon_{e,k} := e^\varepsilon_k|_{B_R\setminus\Omega}$ satisfies the homogeneous equation:

$$\begin{cases} \Delta e^\varepsilon_{e,k} + \kappa^2 q\, e^\varepsilon_{e,k} = 0, & \text{in } B_R \setminus \Omega, \\ \partial_r e^\varepsilon_{e,k} - T_R(e^\varepsilon_{e,k}) = 0, & \text{on } \partial B_R. \end{cases} \tag{2.92}$$

The interior equation. The truncated series $\widetilde{u}^{\varepsilon,k}_\chi$ do not exactly satisfies the same Helmholtz equation inside Ω as the exact solution. They verify this equation with a small right hand side. To see that, let us set:

$$\widetilde{u}^{\varepsilon,k}_{\text{in}} = \sum_{\ell=0}^{k} \varepsilon^\ell\, u^\ell_{\text{in}}, \quad \text{so that,} \quad \widetilde{u}^{\varepsilon,k}_\chi = \chi\,\widetilde{u}^{\varepsilon,k}_{\text{in}} \quad \text{in } \Omega. \tag{2.93}$$

Indeed

$$\Delta\widetilde{u}^{\varepsilon,k}_\chi + \kappa^2_{\text{in}}\,\widetilde{u}^{\varepsilon,k}_\chi + \frac{i}{\varepsilon^2}\,\widetilde{u}^{\varepsilon,k}_\chi = \chi\{\Delta\widetilde{u}^{\varepsilon,k}_{\text{in}} + \kappa^2_{\text{in}}\,\widetilde{u}^{\varepsilon,k}_{\text{in}} + \frac{i}{\varepsilon^2}\widetilde{u}^{\varepsilon,k}_{\text{in}}\} + 2\nabla\chi\cdot\nabla\widetilde{u}^{\varepsilon,k}_{\text{in}} + \Delta\chi\,\widetilde{u}^{\varepsilon,k}_{\text{in}}.$$

Inside the support of χ the local coordinates $(x_\Gamma, \nu = \varepsilon\eta)$ can be used to make the identification (cf. 2.48)

$$\Delta + \kappa_{\rm in}^2 + \frac{i}{\varepsilon^2} \equiv \frac{1}{J_\nu^3 \varepsilon^2} \left(\partial_{\eta\eta}^2 + i + \sum_{\ell=1}^{8} \varepsilon^\ell \mathcal{A}_\ell \right). \tag{2.94}$$

From equation (2.57), after multiplication by the correct power of ε and summation, it is easily seen (the calculations are long but not difficult) that

$$\left(\partial_{\eta\eta}^2 + i + \sum_{\ell=1}^{8} \varepsilon^\ell \mathcal{A}_\ell \right) \widetilde{u}_{\rm in}^{\varepsilon,k} = \varepsilon^{k+1} \sum_{\ell=1}^{8} \sum_{p=0}^{\ell-1} \varepsilon^p \, \mathcal{A}_{\ell-p-1} u_{\rm in}^{k+p+1-\ell}. \tag{2.95}$$

Therefore, thanks to (2.94) and (2.95),

$$\Delta \widetilde{u}_\chi^{\varepsilon,k} + \kappa^2 q_{\rm in} \widetilde{u}_\chi^{\varepsilon,k} + \frac{i}{\varepsilon^2} \, \widetilde{u}_\chi^{\varepsilon,k} = f_{{\rm in},k}^\varepsilon \quad \text{in } \Omega, \tag{2.96}$$

where the function $f_{{\rm in},k}^\varepsilon$ is given by (with obvious notation)

$$g_{{\rm in},k}^\varepsilon = \varepsilon^{k-1} \, \chi \, \sum_{\ell=1}^{8} \sum_{p=0}^{\ell-1} \varepsilon^p \, \mathcal{A}_{\ell-p-1} u_{\rm in}^{k+p+1-\ell}(.,\nu/\varepsilon) + 2\nabla\chi \cdot \nabla \widetilde{u}_{\rm in}^{\varepsilon,k} + \Delta\chi \, \widetilde{u}_{\rm in}^{\varepsilon,k}. \tag{2.97}$$

From expression (2.61) and the identity

$$\int_0^{+\infty} \left(\frac{\nu}{\varepsilon} \right)^\ell e^{-\frac{\nu}{\sqrt{2}\varepsilon}} \, d\nu = C_\ell \, \varepsilon, \quad \forall \, \ell \in \mathbb{N} \,,$$

where C_ℓ is a constant independent of ε, it is not difficult to verify that

$$\left(\int_{\Omega^{\bar\nu}} |u_{\rm in}^\ell(x_\Gamma, \nu/\varepsilon)|^2 \, dx \right)^{\frac{1}{2}} \leq C_\ell(\bar\nu) \, \varepsilon^{\frac{1}{2}} \tag{2.98}$$

for some constant $C_\ell(\bar\nu)$ that depends on $\bar\nu$ and the boundary terms of u_e^k, $k \leq \ell$, but is independent of ε. In the same way, one easily shows that:

$$\left(\int_{\Omega^{\bar\nu} \setminus \Omega^{\frac{\bar\nu}{2}}} \{ \, |\widetilde{u}_{\rm in}^{\varepsilon,k}|^2 + |\nabla \widetilde{u}_{\rm in}^{\varepsilon,k}|^2 \, \} \, dx \right)^{\frac{1}{2}} \leq C(\bar\nu) \, \exp(-\bar\nu/\varepsilon) \,. \tag{2.99}$$

Regrouping estimates (2.98) and (2.99) into (2.97), yields the existence of a constant C independent of ε such that

$$||f_{{\rm in},k}^\varepsilon||_{L^2(\Omega)} \leq C \, \varepsilon^{k-\frac{1}{2}} \,. \tag{2.100}$$

Notice of course that, by taking the difference between (2.96) and (2.3)(ii), $e_{{\rm in},k}^\varepsilon := e_{k|\Omega}^\varepsilon$ satisfies

$$\Delta e_{{\rm in},k}^\varepsilon + (\kappa_{\rm in}^2 + \frac{i}{\varepsilon^2}) e_{{\rm in},k}^\varepsilon = f_{{\rm in},k}^\varepsilon \quad \text{in } \Omega. \tag{2.101}$$

The transmission conditions. From the interface condition (2.59) it is clear that $\widetilde{u}_\chi^{\varepsilon,k}$, and thus e_k^ε, is continuous across Γ. However, from (2.58), due to the shift of index between left and right hand sides, the normal derivative of $\widetilde{u}_\chi^{\varepsilon,k}$ is discontinuous across Γ, so is the normal derivative of e_k^ε). More precisely, straightforward calculations lead to the following transmission conditions

$$\begin{cases} e_{e,k}^\varepsilon - e_{\mathrm{in},k}^\varepsilon = 0, & \text{on } \Gamma, \\ \partial_n\, e_{e,k}^\varepsilon - \partial_n\, e_{\mathrm{in},k}^\varepsilon = \varepsilon^k\, \partial_n\, u_e^k, & \text{on } \Gamma. \end{cases} \tag{2.102}$$

Error estimates. We can now proceed to the final step of the proof. Multiplying equation (2.92) by $\overline{e_{e,k}^\varepsilon}$ and integrating over $B_R \setminus \Omega$, we obtain by using Green's formula,

$$\int_{B_R\setminus\Omega} |\nabla e_{e,k}^\varepsilon|^2\, dx - \kappa^2 \int_{B_R\setminus\Omega} q|e_{e,k}^\varepsilon|^2\, dx - \left\langle T_R(e_{e,k}^\varepsilon), e_{e,k}^\varepsilon \right\rangle_{\partial B_R} = \left\langle \partial_n e_{e,k}^\varepsilon, e_{e,k}^\varepsilon \right\rangle_\Gamma . \tag{2.103}$$

In the same way, multiplying equation (2.101) by $\overline{e_{\mathrm{in},k}^\varepsilon}$ and integrating over Ω, one gets

$$\begin{aligned} \int_\Omega |\nabla e_{\mathrm{in},k}^\varepsilon|^2\, dx - \kappa^2 \int_\Omega q_r |e_{\mathrm{in},k}^\varepsilon|^2\, dx - \frac{i}{\varepsilon^2} \int_\Omega |e_{\mathrm{in},k}^\varepsilon|^2\, dx & \\ = - \left\langle \partial_n e_{e,k}^\varepsilon, e_{e,k}^\varepsilon \right\rangle_\Gamma - \int_\Omega f_{\mathrm{in},k}^\varepsilon\, \overline{e_{\mathrm{in},k}^\varepsilon}\, dx\ . & \end{aligned} \tag{2.104}$$

(recall that by definition, $\kappa_{\mathrm{in}}^2 = \kappa^2 q_r$ inside Ω) Adding together (2.104) and (2.103) and using (2.102) and (2.100), gives (using the properties of T_R and q)

$$\begin{aligned} \left| \int_{B_R} |\nabla e_k^\varepsilon|^2 - \kappa^2 \int_{B_R} q_r |e_k^\varepsilon|^2 - \frac{i}{\varepsilon^2} \int_\Omega |e_k^\varepsilon|^2 \right| & \\ \leq C_k\ \left(\varepsilon^k\ ||e_k^\varepsilon||_{L^2(\Gamma)} + \varepsilon^{k-\frac{1}{2}}\ ||e_k^\varepsilon||_{L^2(\Omega)} \right), & \end{aligned} \tag{2.105}$$

where C_k is a constant independent of ε. Ones deduces the desired error estimates by applying Lemma 2.3. $\square$

2.5.2 Error estimates for the GIBCs

Existence and uniqueness results for the approximate problems. We shall check here that the approximate solutions $u^{\varepsilon,k}$ are well defined. This is our next result.

Lemma 2.4 *For $k = 0, 1, 2, 3$, the boundary value problem:*

$$\begin{cases} \Delta u^{\varepsilon,k} + \kappa^2 q\, u^{\varepsilon,k} = f, & \text{in } B_R \setminus \Omega, \\ \partial_r u^{\varepsilon,k} - T_R(u^{\varepsilon,k}) = g, & \text{on } \partial B_R, \\ u^{\varepsilon,k} + \mathcal{D}^{\varepsilon,k} u^{\varepsilon,k} \partial_n = 0 & \text{on } \Gamma, \end{cases} \tag{2.106}$$

admits a unique solution in $u^{\varepsilon,k} \in H^1(B_R \setminus \Omega)$ provided that $\varepsilon\, \mathcal{H} \leq \sqrt{2}/2$ if $k = 2$ or ε is small enough if $k = 3$.

Proof. Since the proof for $k = 0, 1, 2$ has already been given in Chapter 1, we shall concentrate here on the case $k = 3$. We start by reformulation problem (2.106) as a system.

New formulation of the problem. Introducing $\varphi^\varepsilon = \partial_n u^{\varepsilon,3}|_\Gamma$ as a new unknown, problem (2.106) is equivalent, for $k = 3$, to find $(u^{\varepsilon,3}, \varphi^\varepsilon) \in H^1(B_R \setminus \Omega) \times H^1(\Gamma)$ such that

$$\left| \begin{array}{ll} \Delta u^{\varepsilon,3} + \kappa^2 q\, u^{\varepsilon,3} = f, & \text{in } B_R \setminus \Omega, \\ \partial_r u^{\varepsilon,3} - T_R(u^{\varepsilon,3}) = g, & \text{on } \partial B_R, \\ \partial_n u^{\varepsilon,3} = \varphi^\varepsilon, & \text{on } \Gamma, \\ -\Delta_\Gamma\, \varphi^\varepsilon + \dfrac{2i}{\varepsilon^2}\, \theta_3(\varepsilon)\, \varphi^\varepsilon = -\dfrac{2i\alpha}{\varepsilon^3}\, u^{\varepsilon,3} & \text{on } \Gamma, \end{array} \right. \tag{2.107}$$

where we have set $\theta_3(\varepsilon) = 1 - \dfrac{\varepsilon \mathcal{H}}{\alpha} + i\, \dfrac{\varepsilon^2 A(\kappa_{\text{in}})}{2}$ with $A(\kappa) = 3\mathcal{H}^2 - G + \kappa^2$.

Next we show that problem (2.107) is of Fredholm type. For this, we first notice that (2.107) is equivalent to the variational problem:

$$\left\{ \begin{array}{l} \text{Find } (u^{\varepsilon,3}, \varphi^\varepsilon) \in H^1(B_R \setminus \Omega) \times H^1(\Gamma) \text{ such that } \forall\, (v, \psi) \in H^1(B_R \setminus \Omega) \times H^1(\Gamma), \\ a_1\left((u^{\varepsilon,3}, \varphi^\varepsilon), (v, \psi)\right) + a_2^\varepsilon\left((u^{\varepsilon,3}, \varphi^\varepsilon), (v, \psi)\right) = -\displaystyle\int_{B_R\setminus\Omega} f \overline{v}\, dx + \langle g, v \rangle_{\partial B_R} \end{array} \right. \tag{2.108}$$

where we have set:

$$\begin{array}{rcl} a_1\left((u, \varphi), (v, \psi)\right) & = & \displaystyle\int_{B_R\setminus\Omega} \left(\nabla u \cdot \nabla \overline{v} + u\, \overline{v}\right)\, dx - \langle T_R(u), v \rangle_{\partial B_R} \\ & + & \displaystyle\int_\Gamma (\nabla_\Gamma\, \varphi \cdot \nabla_\Gamma\, \overline{\phi} + \varphi\, \overline{\psi})\, ds \\ a_2^\varepsilon\left((u, \varphi), (v, \psi)\right) & = & -\displaystyle\int_{B_R\setminus\Omega} (\kappa^2 q + 1) u\, \overline{v}\, dx - \int_\Gamma [\, 1 - \frac{2i}{\varepsilon^2}\, \theta_3(\varepsilon)\,] \varphi\, \overline{\psi}\, ds \\ & + & \dfrac{2i\alpha}{\varepsilon^3} \displaystyle\int_\Gamma u\, \overline{\psi}\, ds\, - \int_\Gamma \varphi\, \overline{v}\, ds. \end{array}$$

One next remarks that $a_1(\cdot, \cdot)$ is coercive in $H^1(B_R \setminus \Omega) \times H^1(\Gamma)$ while $a_2^\varepsilon(\cdot, \cdot)$ is weakly compact in $H^1(B_R \setminus \Omega) \times H^1(\Gamma)$:

$$(u^n, \varphi^n) \rightharpoonup (u, \varphi) \text{ in } H^1(B_R \setminus \Omega) \times H^1(\Gamma) \Longrightarrow a_2^\varepsilon\left((u^n, \varphi^n)(u^n, \varphi^n)\right) \to a_2^\varepsilon\left((u, \varphi)(u, \varphi)\right).$$

Therefore, formulation (2.108) satisfies the Fredholm alternative: existence and uniqueness of solutions is equivalent to uniqueness of solutions to(2.107) (or (2.108)).

Uniqueness proof. We prove the uniqueness result for ε small enough by contradiction. If uniqueness fails then, up to the extraction of a sequence of values of ε tending to 0, one can assume that here exists a non trivial solution $(u^{\varepsilon,3}, \varphi^{\varepsilon})$ of the homogeneous problem associated with (2.107), that we can normalize in such a way that:

$$\|u^{\varepsilon,3}\|_{L^2(B_R\setminus\Omega)} = 1. \tag{2.109}$$

We multiply the Helmholtz equation by the complex conjugate of $u^{\varepsilon,3}$ and after integration by parts, we replace, in the boundary term on Γ, the trace of $u^{\varepsilon,3}$ by its expression as a function of φ^{ε} from the last equation of (2.107). This leads to

$$\int_{B_R\setminus\Omega} \left(|\nabla u^{\varepsilon,3}|^2 - \kappa^2 q|u^{\varepsilon,3}|^2\right)\, dx + \frac{\bar{\alpha}\,\varepsilon^3}{2}\int_\Gamma |\nabla_\Gamma\, \varphi^\varepsilon|^2\, ds \\ + \varepsilon\alpha \int_\Gamma \theta_3(\varepsilon)\, |\varphi^\varepsilon|^2\, ds - \langle T_R(u), u\rangle_{\partial B_R} = 0.$$

We now take the real part of the last equality (contrary to what is more usual, taking the imaginary part does not provide the desired estimate, the term in $|\nabla_\Gamma\, \varphi^\varepsilon|^2$ comes with the wrong sign) and use (2.109) to get that

$$\int_{B_R\setminus\Omega} |\nabla u^{\varepsilon,3}|^2\, dx + \frac{\varepsilon^3\sqrt{2}}{4}\int_\Gamma |\nabla_\Gamma\, \varphi^\varepsilon|^2\, ds + \varepsilon \int_\Gamma \mathcal{R}e(\alpha\theta_3(\varepsilon))|\varphi^\varepsilon|^2\, ds \le C, \tag{2.110}$$

where C is a constant independent of ε.
Since $\mathcal{R}e(\alpha\theta_3(\varepsilon))$ tends to $\sqrt{2}/2$ as ε goes to 0, we deduce from previous estimate that $u^{\varepsilon,3}$ is bounded in $H^1(B_R\setminus\Omega)$. Therefore, up to the extraction of a subsequence, we can assume that:

$$\begin{cases} u^{\varepsilon,3} \to u, & \text{weakly in } H^1(B_R\setminus\Omega), \\ u^{\varepsilon,3} \to u, & \text{strongly in } L^2(B_R\setminus\Omega), \\ \Delta u^{\varepsilon,3} \to \Delta u, & \text{weakly in } L^2(B_R\setminus\Omega), \end{cases}$$

the latter property being deduced from the Helmholtz equation. By trace theorem, $\partial_n u^{\varepsilon,3}|_\Gamma$ (resp. $\partial_n u^{\varepsilon,3}|_{\partial B_R}$) converges to $\partial_n u|_\Gamma$ (resp. $\partial_n u|_{\partial B_R}$) in $H^{-\frac{1}{2}}(\Gamma)$ (resp. $H^{-\frac{1}{2}}(\partial B_R)$). Of course, at the limit, we have:

$$\begin{cases} \Delta u + \kappa^2 q\, u = 0, & \text{in } B_R\setminus\Omega, \\ \partial_r u - T_R(u) = 0, & \text{in } \partial B_R. \end{cases} \tag{2.111}$$

while, passing to the (weak) limit in the last boundary equation of (2.107) after multiplication by ε^3, we obtain

$$u = 0, \qquad \text{on } \Gamma. \tag{2.112}$$

From Lemma 1.3 we then obtain that $u = 0$ in $B_R\setminus\Omega$, which is in contradiction with $\|u\|_{L^2(B_R\setminus\Omega)} = 1$. □

Analysis of the difference $u^{\varepsilon,k} - \widetilde{u}^{\varepsilon,k}$. We now proceed to step 2 of the sketch announced in Section 3. From now on, we shall set for $k = 0, 1, 2, 3$,

$$\mathbf{e}^{\varepsilon,k} = u^{\varepsilon,k} - \widetilde{u}^{\varepsilon,k}. \tag{2.113}$$

The starting point of the error analysis is to remark that $\mathbf{e}^{\varepsilon,k}$ is a solution of a homogeneous Helmholtz equation with outgoing absorbing condition on ∂B_R,

$$\begin{cases} \Delta \mathbf{e}^{\varepsilon,k} + \kappa^2 q \, \mathbf{e}^{\varepsilon,k} = 0 \quad \text{in} \quad B_R \setminus \Omega, \\ \partial_r \mathbf{e}^{\varepsilon,k} - T_R(\mathbf{e}^{\varepsilon,k}) = 0 \quad \text{on} \quad \partial B_R, \end{cases} \tag{2.114}$$

and satisfies a non homogeneous *GIBC* boundary condition on Γ with small right hand side. This comes directly from the construction of the *GIBC* itself and is obtained by making the difference between (2.70) and (2.16). Let us formulate this as a lemma:

Lemma 2.5 *For $k = 1, 2, 3$, there exists a smooth function g_k^ε such that*

$$\mathbf{e}^{\varepsilon,k} + \mathcal{D}^{\varepsilon,k} \, \partial_n \mathbf{e}^{\varepsilon,k} = \varepsilon^{k+1} \, g_k^\varepsilon, \tag{2.115}$$

and such that

$$||g_k^\varepsilon||_{H^{\frac{1}{2}}(\Gamma)} \leq C_k, \quad \textit{for } k = 1, 2, 3, \tag{2.116}$$

where C_k is a positive constant independent of ε.

This result can be seen as a *consistency* result for the boundary condition. Combined with a *stability* argument, it is then possible to obtain the following estimates.

Lemma 2.6 *For $k = 1, 2, 3$, there exists a positive constant C_k independent of ε such that*

$$\|u^{\varepsilon,k} - \widetilde{u}^{\varepsilon,k}\|_{H^1(B_R \setminus \Omega)} \leq \; C_k \, \varepsilon^{k+1}. \tag{2.117}$$

Proof. From (2.114) and (2.115) and Green's formula,

$$\left| \begin{array}{l} \displaystyle\int_{B_R \setminus \Omega} (\, |\nabla \mathbf{e}^{\varepsilon,k}|^2 - \kappa^2 q |\mathbf{e}^{\varepsilon,k}|^2 \,) \, dx - \left\langle T_R(\mathbf{e}^{\varepsilon,k}), \mathbf{e}^{\varepsilon,k} \right\rangle_{\partial B_R} \\ \qquad \displaystyle\int_\Gamma \mathcal{D}^{\varepsilon,k} \, \partial_n \mathbf{e}^{\varepsilon,k} \cdot \overline{\partial_n \mathbf{e}^{\varepsilon,k}} \, ds = \varepsilon^{k+1} \int_\Gamma g_k^\varepsilon \, \overline{\partial_n \mathbf{e}^{\varepsilon,k}} \, ds. \end{array} \right. \tag{2.118}$$

Setting $\varphi_k^\varepsilon = \partial_n \mathbf{e}^{\varepsilon,k}|_\Gamma$, and introducing the functions $\theta_1(\varepsilon) = 1$, $\theta_2(\varepsilon) = 1 - \frac{\varepsilon \mathcal{H}}{\alpha}$, ($\theta_3(\varepsilon)$ has been defined in the proof of Lemma 2.4), one can derive the following general identity by using the explicit expression of $\mathcal{D}^{\varepsilon,k}$,

$$\left| \begin{array}{l} \displaystyle\int_{B_R \setminus \Omega} (\, |\nabla \mathbf{e}^{\varepsilon,k}|^2 - \kappa^2 q |\mathbf{e}^{\varepsilon,k}|^2 \,) \, dx - \left\langle T_R(\mathbf{e}^{\varepsilon,k}), \mathbf{e}^{\varepsilon,k} \right\rangle_{\partial B_R} \\ \quad \displaystyle + \, \varepsilon \alpha \int_\Gamma \theta_k(\varepsilon) \, |\varphi_k^\varepsilon|^2 \, ds + \nu_k \, \frac{\bar{\alpha} \, \varepsilon^3}{2} \int_\Gamma |\nabla_\Gamma \, \varphi_k^\varepsilon|^2 \, ds = \varepsilon^{k+1} \int_\Gamma g_k^\varepsilon \, \overline{\partial_n \mathbf{e}^{\varepsilon,k}} \, ds \; , \end{array} \right. \tag{2.119}$$

where $\nu_k = 0$ for $k = 0, 1, 2$ and $\nu_3 = 1$. Taking the real part,

$$\int_{B_R\setminus\Omega} (\ |\nabla \mathbf{e}^{\varepsilon,k}|^2 - \kappa^2 q_r |\mathbf{e}^{\varepsilon,k}|^2\ )\ dx - \mathcal{R}e \left\langle T_R(\mathbf{e}^{\varepsilon,k}), \mathbf{e}^{\varepsilon,k} \right\rangle_{\partial B_R} +\ \varepsilon \int_\Gamma \mathcal{R}e(\alpha\theta_k(\varepsilon))\ |\varphi_k^\varepsilon|^2\ ds$$
$$+\ \nu_k\ \frac{\sqrt{2}\ \varepsilon^3}{4} \int_\Gamma |\nabla_\Gamma\ \varphi_k^\varepsilon|^2\ ds = \varepsilon^{k+1}\ \mathcal{R}e \int_\Gamma g_k^\varepsilon\ \overline{\partial_n \mathbf{e}^{\varepsilon,k}}\ ds\ . \tag{2.120}$$

In particular, since $\nu_k \geq 0$ and $\mathcal{R}e(\alpha\theta_k(\varepsilon))$ tends to $\sqrt{2}$ as ε tends to 0, we obtain the following estimate, for ε small enough,

$$\left| \int_{B_R\setminus\Omega} (\ |\nabla \mathbf{e}^{\varepsilon,k}|^2 - \kappa^2 q_r |\mathbf{e}^{\varepsilon,k}|^2\ )\ dx \right| \begin{array}{l} \leq\ \varepsilon^{k+1}\ \|g_k^\varepsilon\|_{H^{\frac{1}{2}}(\Gamma)} \|\partial_n \mathbf{e}^{\varepsilon,k}\|_{H^{-\frac{1}{2}}(\Gamma)} \\ \leq\ C_k\ \varepsilon^{k+1}\ \|\mathbf{e}^{\varepsilon,k}\|_{H^1(B_R)}, \end{array} \tag{2.121}$$

where the latter inequality comes from (2.116) and the fact that $\mathbf{e}^{\varepsilon,k}$ is solution the Helmholtz equation inside $B_R \setminus \Omega$. The remaining part of the proof in then rather straightforward. We first prove by contradiction that

$$\|\mathbf{e}^{\varepsilon,k}\|_{L^2(B_R\setminus\Omega)} \leq C_k\ \varepsilon^{k+1}\ . \tag{2.122}$$

If (2.122) is not true, then $\mu_k^\varepsilon = \varepsilon^{-(k+1)} \|\mathbf{e}^{\varepsilon,k}\|$ would blows up (for a subsequence) as ε goes to 0. Then, introducing

$$w^{\varepsilon,k} = \mathbf{e}^{\varepsilon,k}\ /\ \|\mathbf{e}^{\varepsilon,k}\|_{L^2(B_R\setminus\Omega)},$$

ones derives from (2.121)

$$\int_{B_R\setminus\Omega} |\nabla w^{\varepsilon,k}|^2\ dx \leq \kappa^2 + C_k\ (\mu_k^\varepsilon)^{-1}\ \|w^{\varepsilon,k}\|_{H^1(B_R)} \leq C_k\ (1 + \|w^{\varepsilon,k}\|_{H^1(B_R)}). \tag{2.123}$$

Therefore, $w^{\varepsilon,k}$ is bounded in $H^1(B_R)$ and thus, up to the extraction of a subsequence, converges weakly in $H^1(B_R \setminus \Omega)$ but strongly in $L^2(B_R \setminus \Omega)$ to some $w^k \in H^1(B_R \setminus \Omega)$ that satisfies $\|w^k\|_{L^2(B_R\setminus\Omega)} = 1$ as well as

$$\left\{ \begin{array}{ll} \Delta w^k + \kappa^2 w^k = 0, & \text{in}\ \ B_R \setminus \Omega, \\ \partial_r w^k - T_R(\kappa w^k) = 0, & \text{on}\ \ \partial B_R. \end{array} \right. \tag{2.124}$$

Finally, passing to the limit (in the weak sense) in the boundary condition

$$w^{\varepsilon,k} + \mathcal{D}^{\varepsilon,k}\ \partial_n w^{\varepsilon,k} = g_k^\varepsilon /\ \|\mathbf{e}^{\varepsilon,k}\|_{L^2(B_R\setminus\Omega)} =\ (\mu_k^\varepsilon)^{-1}\ (g_k^\varepsilon /\ \varepsilon^{k+1})\ , \tag{2.125}$$

we see ($g_k^\varepsilon / \varepsilon^{k+1}$ is bounded and $(\mu_k^\varepsilon)^{-1}$ tends to 0) that w^k also satisfies

$$w^k = 0, \quad \text{on}\ \Gamma. \tag{2.126}$$

System ((2.124),(2.126)) implies that $w^k = 0$, which contradicts $\|w^k\|_{L^2(B_R)} = 1$. Therefore, (2.122) holds. The Lemma estimate is now a direct consequence of (2.122) and (2.121). □

2.6 About the analysis of modified GIBCs

The error analysis of modified $GIBC$s can be done in a similar way as for the NtD $GIBC$s. We shall restrict ourselves to stating the results and indicating the needed modifications in previous section proofs.

2.6.1 Analysis of DtN $GIBC$s

Theorem 2.4 *Let $k = 1, 2$ or 3, then, assuming ε beeing sufficiently small when $k = 3$, the boundary value problem ((2.12), (2.25)) has a unique solution $u^{\varepsilon,k} \in H^1(B_R \setminus \Omega)$. Moreover, there exists a constant C_k, independent of ε, such that*

$$\|u_e^\varepsilon - u^{\varepsilon,k}\|_{H^1(B_R\setminus\Omega)} \leq C_k\ \varepsilon^{k+1}. \tag{2.127}$$

Proof. We shall only treat here the case $k = 3$ (the others are easy) and start with the proof of estimate (2.127) assuming the existence and uniqueness of the solution (whose verification is left as an exercise).

Of course, we only need to consider the difference $u^{\varepsilon,3} - \tilde{u}^{\varepsilon,3}$, namely to prove the equivalent to Lemma 2.6.

Rather curiously, it appears that treating the boundary condition directly in its DtN form (2.25) does not lead immediately to the optimal error estimate. This is why we shall rewrite it as an NtD condition by introducing the inverse of the operator $\mathcal{N}^{\varepsilon,3}$ (note that, by Lax-Milgram's lemma, $\mathcal{N}^{\varepsilon,3}$ is an isomorphism from $H^{s+2}(\Gamma)$ onto $H^s(\Gamma)$).

We repeat here the approach of Lemma 2.6. One first checks that the error $\mathbf{e}^{\varepsilon,3}$ satisfies the homogeneous Helmholtz equation in $B_R \setminus \Omega$ together with the non homogeneous boundary condition:

$$\mathbf{e}^{\varepsilon,3} + \left(\mathcal{N}^{\varepsilon,3}\right)^{-1} \partial_n \mathbf{e}^{\varepsilon,3} = \varepsilon^4\ g_3^\varepsilon, \tag{2.128}$$

where g_3^ε is a smooth function satisfying:

$$||g_3^\varepsilon||_{H^{\frac{1}{2}}(\Gamma)} \leq C_k, \quad \text{for } k = 1, 2, 3, \tag{2.129}$$

Proceeding as in the proof of Lemma 2.6, we obviously get

$$\left| \begin{aligned} &\int_{B_R\setminus\Omega} (\ |\nabla \mathbf{e}^{\varepsilon,k}|^2 - \kappa^2 q |\mathbf{e}^{\varepsilon,3}|^2\ )\ dx - \left\langle T_R(\mathbf{e}^{\varepsilon,3}), \mathbf{e}^{\varepsilon,k} \right\rangle_{\partial B_R} \\ &\qquad + \int_\Gamma \overline{\left(\mathcal{N}^{\varepsilon,3}\right)^{-1} \partial_n \mathbf{e}^{\varepsilon,3}} \cdot \partial_n \mathbf{e}^{\varepsilon,3}\ ds = \varepsilon^{k+1} \int_\Gamma g_3^\varepsilon\ \overline{\partial_n \mathbf{e}^{\varepsilon,3}}\ ds. \end{aligned} \right. \tag{2.130}$$

The key point is that, at least for ε small enough, for any ψ smooth enough,

$$\mathcal{R}e\ \int_\Gamma \overline{\left(\mathcal{N}^{\varepsilon,3}\right)^{-1} \psi} \cdot \psi\ dx \leq 0. \tag{2.131}$$

This is a consequence of

$$\mathcal{R}e \int_\Gamma \mathcal{N}^{\varepsilon,3}\varphi \cdot \overline{\varphi}\, dx \leq 0, \qquad \text{for any } \varphi \text{ smooth enough,}$$

that follows from the identity (proven in section 2.3.2)

$$\int_\Gamma \varphi \cdot \overline{\mathcal{N}^{\varepsilon,3}\varphi}\, ds = \frac{\alpha\,\varepsilon}{2} \int_\Gamma |\nabla_\Gamma \varphi|^2\, ds + \frac{\bar{\alpha}}{\varepsilon} \int_\Gamma [\, 1 + \frac{\varepsilon \mathcal{H}}{\bar{\alpha}} - i\, \frac{\varepsilon^2}{2}\, (\mathcal{H}^2 - G + \kappa^2)\,]\, |\varphi|^2\, ds,$$

and the observation that

$$\begin{cases} \mathcal{R}e\ \alpha = \mathcal{R}e\ \bar{\alpha} = \sqrt{2}/2, \\[2ex] \displaystyle\lim_{\varepsilon \to 0} [\, 1 + \frac{\varepsilon \mathcal{H}}{\bar{\alpha}} - i\, \frac{\varepsilon^2}{2}\, (\mathcal{H}^2 - G + \kappa^2)\,] = 1. \end{cases}$$

Therefore we have shown that, as soon as ε is small enough,

$$\int_{B_R \setminus \Omega} (\, |\nabla \mathbf{e}^{\varepsilon,k}|^2 - \kappa^2 q_r |\mathbf{e}^{\varepsilon,3}|^2\,)\, dx \leq 0, \tag{2.132}$$

and the conclusion of the proof is identical to the one of Lemma 2.6. $\square$.

Remark 2.13 *Proceeding as in Section 2.4.4 (for deriving formulas (2.71)), one first gets:*

$$\partial_n \mathbf{e}^{\varepsilon,3} + \mathcal{N}^{\varepsilon,3}\mathbf{e}^{\varepsilon,3} = \varepsilon^3\, h_3^\varepsilon,$$

where h_3^ε (as g_3^ε in formula (2.71)) depends polynomially with respect to ε. It is a polynomial of degree 3 whose coefficients are smooth functions of x_Γ, that can be explicitly expressed in terms of u_e^1, u_e^2 and u_e^3. In particular:

$$h_3^\varepsilon = O(1), \quad \text{in any Sobolev norm.}$$

One then deduces (2.128) with:

$$g_3^\varepsilon = (\varepsilon \mathcal{N}^{\varepsilon,3})^{-1}\ h_3^\varepsilon\ .$$

One finally obtains (2.129) after having noticed that (cf (2.28):

$$(\varepsilon \mathcal{N}^{\varepsilon,3})^{-1} = \frac{1}{\alpha}\ \left\{ 1 + \frac{\varepsilon}{\alpha}\mathcal{H} - i\frac{\varepsilon^2}{2}(\Delta_\Gamma + \mathcal{H}^2 - G + \kappa^2) \right\}^{-1} = O(\varepsilon).$$

2.6.2 Analysis of robust $GIBC$s

Theorem 2.5 *For any $\varepsilon > 0$, the boundary value problem associated with (2.12) and the boundary condition:*

$$u^{\varepsilon,3} + \mathcal{D}_r^{\varepsilon,3}\partial_n u^{\varepsilon,3} = 0, \qquad \text{on } \Gamma, \tag{2.133}$$

where $\mathcal{D}_r^{\varepsilon,3}$ is given by (2.31), has a unique solution $u^{\varepsilon,3} \in H^1(B_R \setminus \Omega)$. Moreover, there exists a constant C_3, independent of ε, such that

$$\|u_e^\varepsilon - u^{\varepsilon,3}\|_{H^1(B_R\setminus\Omega)} \leq C_3\, \varepsilon^4. \tag{2.134}$$

The same result holds if one replaces (2.133) by:

$$\partial_n u^{\varepsilon,3} + \mathcal{N}_r^{\varepsilon,3} u^{\varepsilon,3} = 0, \qquad \text{on } \Gamma, \tag{2.135}$$

where $\mathcal{N}_r^{\varepsilon,3}$ is given by (2.36).

The proof of this theorem is almost identical to the one of Theorem 2.4 or Lemma 2.6 and is left as an exercise. Let us simply indicate that the existence and uniqueness result is valid for any positive ε due to properties (2.34) and (2.37). The main difference lies in the fact that the algebra to obtain the equivalent to identities (2.5) and (2.128) is slightly more complicated and the calculations for obtaining the equivalent to property (2.131) are longer.

Chapter 3

Rudiments of differential geometry

We recall in this appendix some well known facts on differential geometry and some results in connection with surface operators. To simplify the presentation we shall consider C^2 surfaces Γ that can be extended to a boundary of a simply connected domain and we denote by n a C^1 unitary normal field defined on Γ (this implies in particular that the surface Γ is located on one side of its normal n).

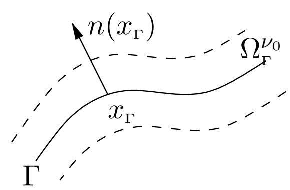
Local coordinates. Let $\bar{\nu}$ be a positive real. We define the tubular neighborhood $\Omega_\Gamma^{\bar{\nu}}$ of Γ by

$$\Omega_\Gamma^{\bar{\nu}} := \{x \in \mathbb{R}^3\,; x = x_\Gamma + \nu\, n(x_\Gamma) \ \text{ for some } (x_\Gamma, \nu) \in \Gamma \times (-\bar{\nu}, \bar{\nu})\}.$$

Then, for a sufficiently small positive constant ν_0 (see condition (3.4) below) the application $(x_\Gamma, \nu) \mapsto x$ defined by

$$x = x_\Gamma + \nu\, n(x_\Gamma) \tag{3.1}$$

is a bijection from $\Gamma \times (-\nu_0, \nu_0)$ onto $\Omega_\Gamma^{\nu_0}$. The couple (x_Γ, ν) defined by (3.1) is denoted by the parametric local coordinates of x with respect to Γ. Notice that x_Γ is nothing but the orthogonal projection of x on Γ, i.e. x_Γ is the point on Γ that minimizes $y \mapsto |x - y|$ for $y \in \Gamma$.



Tangential (or surface) differential operators. In what follows we deal with various fields defined on Γ: scalar fields φ (with values in $\mathbb{C}$), vector fields V (with values in $\mathbb{C}^3$) and matrix (or tensor) fields $\mathbf{A}$ (with values in $\mathcal{L}(\mathbb{C}^3)$). By definition:

- A vector field V is tangential if and only if $V \cdot n = 0$ (as a scalar field along Γ).
- A matrix field $\mathbf{A}$ is tangential if and only if $\mathbf{A}\, n = 0$ (as a vector field along Γ).

For simplicity, we assume that these fields have at least C^1 regularity, but this can be removed by interpreting the derivatives in the sense of distributions.

We recall that the surface gradient operator ∇_Γ is defined by:

$$\nabla_\Gamma\, \varphi(x_\Gamma) = \nabla\hat{\varphi}(x_\Gamma), \quad \forall \varphi : \Gamma \to \mathbb{R},$$

where $\hat{\varphi}$ is the 3-D vector field defined locally in $\Omega_\Gamma^{\nu_0}$ by $\hat{\varphi}(x_\Gamma + \nu\, n) = \varphi(x_\Gamma)$. Note that $\nabla_\Gamma\, \varphi$ is a tangential vector field. We can define in the same way the surface gradient of a vector field as a tangential matrix field whose columns are the surface gradients of each component of the vector field.

We denote by $-\mathrm{div}_\Gamma$ the $L^2(\Gamma)$- adjoint of ∇_Γ : $-\mathrm{div}_\Gamma$ maps a tangential vector field into a scalar field. More generally, if $\mathbf{A}(x_\Gamma)$ is a tangential matrix field on Γ, we define the operator $\mathbf{A}\nabla_\Gamma$ for a scalar field $\varphi(x_\Gamma)$ by

$$(\mathbf{A}\nabla_\Gamma\,)u := \mathbf{A}(\nabla_\Gamma\, u).$$

In the same way, we define the operator $(\mathbf{A}\nabla_\Gamma\,)\cdot$ acting on a tangential vector field $V(x_\Gamma)$ as:

$$(\mathbf{A}\nabla_\Gamma\,) \cdot V := \sum_{i=1}^{3} (\mathbf{A}\nabla_\Gamma\, V_i)_i,$$

where the subscript i denotes the i^{th} component of a vector in the canonical basis of $\mathbb{R}^3$.

We then define the surface curl of a tangential vector filed $V(x_\Gamma)$ and the surface vector curl of a scalar function $\varphi(x_\Gamma)$ as

$$\mathrm{curl}_\Gamma\, V := \mathrm{div}_\Gamma\, (V \times n) \quad \text{and} \quad \vec{\mathrm{curl}}_\Gamma\, \varphi := (\nabla_\Gamma\, \varphi) \times n.$$

Of course, the various operators ∇_Γ, div_Γ, $\mathbf{A}\nabla_\Gamma$, curl_Γ and $\vec{\mathrm{curl}}_\Gamma$ applies in principle to functions defined on Γ. However, they can obviously be understood as (partial) differential operators acting on fields defined in the (3-D) domain $\Gamma \times (-\nu_0, \nu_0)$. For instance, if $\phi \in C^1(\Gamma \times (-\nu_0, \nu_0))$, we define $\nabla_\Gamma\, \phi \in C^0(\Gamma \times (-\nu_0, \nu_0))^3$ as follows (with obvious notation):

$$\forall\, \nu \in (-\nu_0, \nu_0), \quad \Big[\nabla_\Gamma\, \phi\Big](\cdot, \nu) := \nabla_\Gamma\, \Big[\phi(\cdot, \nu)\Big].$$

We apply similar rules to div_Γ, $\mathbf{A}\nabla_\Gamma$, curl_Γ and $\vec{\mathrm{curl}}_\Gamma$. The extension of these definition in the sense of distributions is also elementary (as soon as Γ is C^∞).

Geometrical tools. In what follows, and for the sake of the notation conciseness, we shall most of time not explicitly indicate the dependence on x_Γ of the functions, except when we feel it necessary. We shall be more precise in mentioning the possible dependence with respect to the normal coordinate ν.

A particularly fundamental tensor field is the curvature tensor $\mathcal{C}$, defined by $\mathcal{C} := \nabla_\Gamma n$. We recall that $\mathcal{C}$ is symmetric and $\mathcal{C}\, n = 0$. We denote c_1, c_2 the other two eigenvalues of $\mathcal{C}$ (namely the *principal curvatures*) associated with tangential eigenvectors τ_1, τ_2 ($\tau_1 \cdot n = \tau_2 \cdot n = 0$). We also introduce

$$g := c_1 c_2 \quad \text{and} \quad h := \frac{1}{2}(c_1 + c_2) \tag{3.2}$$

which are respectively the *Gaussian* and *mean curvatures* of Γ, and also introduce the associated matrix fields:

$$\mathcal{H} = h\, I_\Gamma \quad \text{and} \quad \mathcal{G} = g\, I_\Gamma, \tag{3.3}$$

where $I_\Gamma(x_\Gamma)$ denotes the projection operator on the tangent plane to Γ at x_Γ.

Let us introduce (this is the Jacobian of the transformation $(x_\Gamma, \nu) \to x$ - see (3.1))

$$J(\nu)\ \big(= J(\nu, x_\Gamma)\big)\ := \det(I + \nu\, \mathcal{C}) = 1 + 2\nu h + \nu^2 g,$$

and we choose ν_0 sufficiently small in such a way that

$$\forall\, |\nu| < \nu_0, \quad \forall\, x_\Gamma \in \Gamma, \quad J(\nu, x_\Gamma) = 1 + 2\nu h(x_\Gamma) + \nu^2 g(x_\Gamma) > 0. \tag{3.4}$$

Thus, for $|\nu| < \nu_0$, there exists a tangential matrix field $x_\Gamma \to \mathcal{R}_\nu(x_\Gamma)$ such that

$$(I + \nu\, \mathcal{C}(x_\Gamma))\, \mathcal{R}_\nu(x_\Gamma) = I_\Gamma(x_\Gamma).$$

More precisely, there exists a tangential matrix field on Γ, $\mathcal{M}(x_\Gamma)$, such that:

$$I_\Gamma + \nu \mathcal{M} := J(\nu)\, \mathcal{R}_\nu, \quad \forall\, x_\Gamma \in \Gamma, \quad \forall\, |\nu| < \nu_0. \tag{3.5}$$

One easily sees (using for instance the eigenbasis (τ_1, τ_2, n) of $\mathcal{C}$) that

$$\mathcal{M} = 2\mathcal{H} - \mathcal{C} \quad \text{and} \quad \mathcal{M}\, \mathcal{C} = \mathcal{G}.$$

Expression of common derivative operators in local coordinates We refer to [17] for a detailed proof of the following expressions.

The gradient operator Let v be a regular scalar function defined in $\Omega_\Gamma^{\nu_0}$ and $\tilde{v}$ the function defined on $\Gamma \times (-\nu_0, \nu_0)$ by

$$\tilde{v}(x_\Gamma, \nu) = v(x_\Gamma + \nu n),$$

then

$$\nabla v = \mathcal{R}_\nu \nabla_\Gamma\, \tilde{v} + \partial_\nu \tilde{v}\; n. \tag{3.6}$$

where ∇_Γ is the surface gradient on Γ. If one sets

$$J_\nu := \det(I + \nu\, \mathcal{C}) = 1 + 2\nu \mathcal{H} + \nu^2 G,$$

The Laplace operator The following expression of the Laplace operator can be easily deduced from the expression of the gradient operator and the Green formula:

$$\Delta v = \frac{1}{J_\nu} \mathrm{div}_\Gamma\, (\mathcal{R}_\nu J_\nu \mathcal{R}_\nu) \nabla_\Gamma\, \tilde{v} + \frac{1}{J_\nu} \partial_\nu J_\nu \partial_\nu \tilde{v}. \tag{3.7}$$

Using (3.5), this expression can also be written in the form,

$$\Delta v = \frac{1}{J_\nu}\, \mathrm{div}_\Gamma\, \left(\frac{1}{J_\nu} (I_\Gamma + \nu \mathcal{M})^2 \right) \nabla_\Gamma\, \tilde{v} + \frac{1}{J_\nu} \partial_\nu J_\nu \partial_\nu \tilde{v}.$$

or, also in the equivalent form

$$\begin{aligned} J_\nu^3\, \Delta v \quad &= \quad J_\nu\, \mathrm{div}_\Gamma\, (I_\Gamma + \nu \mathcal{M})^2 \nabla_\Gamma\, \tilde{v} - (\nabla_\Gamma\, J_\nu) \cdot (I_\Gamma + \nu \mathcal{M})^2 \nabla_\Gamma\, \tilde{v} \\ &\quad + 2 J_\nu^2\, (\mathcal{H} + \nu G)\, \partial_\nu \tilde{v} + J_\nu^3\, \partial_{\nu\nu}^2 \tilde{v}. \end{aligned} \tag{3.8}$$

The latter expression would be more convenient for the asymptotic matching procedure used in deriving GIBCs, since the right hand side coefficients depend polynomially on the normal coordinate ν.

The curl operator The curl of a regular 3-D vector field $V : \Omega_i^{\nu_0} \to \mathbb{R}^3$ is given in parametric coordinates by:

$$\mathrm{curl}\; V = \Big[(\mathcal{R}_\nu \nabla_\Gamma\,) \cdot (\tilde{V} \times n) \Big]\, n + \Big[\mathcal{R}_\nu \nabla_\Gamma\, (\tilde{V} \cdot n) \Big] \times n - (\mathcal{R}_\nu \mathcal{C} \tilde{V}) \times n - \partial_\nu (\tilde{V} \times n),$$

where V and $\tilde{V}$ (defined on $\Gamma \times (-\nu_0, \nu_0)$) are related by

$$\tilde{V}(x_\Gamma, \nu) = V(x_\Gamma + \nu\, n).$$

As for the Laplace operator, this formula can be written in a more convenient form, after multiplication by $J(\nu)$:

$$\left| \begin{aligned} J(\nu)\; \mathrm{curl}\; V \quad &= \quad \Big[\big((I + \nu \mathcal{M}) \nabla_\Gamma\, \big) \cdot (\tilde{V} \times n) \Big]\, n + \Big[(I + \nu \mathcal{M}) \nabla_\Gamma\, (\tilde{V} \cdot n) \Big] \times n \\ &- \quad \Big[(\mathcal{C} + \nu \mathcal{G}) \tilde{V} \Big] \times n - J(\nu)\; \partial_\nu (\tilde{V} \times n), \end{aligned} \right.$$

or, in an equivalent form,

$$J(\nu)\; \mathrm{curl}\; V = \big(C_\Gamma + \nu\; C_\Gamma^M \big)\, \tilde{V} - J(\nu)\; \partial_\nu (\tilde{V} \times n) \tag{3.9}$$

where we have introduced the notation

$$\left\{ \begin{aligned} & C_\Gamma \tilde{V} = \big(\mathrm{curl}_\Gamma\, \tilde{V} \big) n + \vec{\mathrm{curl}}_\Gamma\, (\tilde{V} \cdot n) - \mathcal{C} \tilde{V} \times n \\ & C_\Gamma^M \tilde{V} = \big(\mathrm{curl}_\Gamma^M\, \tilde{V} \big) n + \vec{\mathrm{curl}}_\Gamma^M\, (\tilde{V} \cdot n) - \mathcal{G} \tilde{V} \times n \\ & \vec{\mathrm{curl}}_\Gamma^M\, u := (\mathcal{M} \nabla_\Gamma\, u) \times n \quad \text{and} \quad \mathrm{curl}_\Gamma^M\, \tilde{V} = (\mathcal{M} \nabla_\Gamma\,) \cdot (\tilde{V} \times n). \end{aligned} \right. \tag{3.10}$$

This is the expression used in the asymptotic matching procedure when deriving GIBCs for electromagnetic problems.

Bibliography

[1] T. Abboud and H. Ammari. Diffraction at a curved grating: TM and TE cases, homogenization. *J. Math. Anal. Appl.*, 202(3):995–1026, 1996.

[2] H. Ammari and C. Latiri-Grouz. Conditions aux limites approchées pour les couches minces périodiques. *M2AN Math. Model. Numer. Anal.*, 33(4):673–693, 1999.

[3] X. Antoine and H. Barucq. Microlocal diagonalization of strictly hyperbolic pseudodifferential systems and application to the design of radiation conditions in electromagnetism. *SIAM J. Appl. Math.*, 61(6):1877–1905, 2001.

[4] X. Antoine, H. Barucq, and A. Bendali. Bayliss-Turkel-like radiation conditions on surfaces of arbitrary shape. *J. Math. Anal. Appl.*, 229(1):184–211, 1999.

[5] X. Antoine, H. Barucq, and L. Vernhet. High-frequency asymptotic analysis of a dissipative transmission problem resulting in generalized impedance boundary conditions. *Asymptot. Anal.*, 26(3-4):257–283, 2001.

[6] M. Artola and M. Cessenat. Diffraction d'une onde électromagnétique par une couche composite mince accolée à un corps conducteur épais. I. Cas des inclusions fortement conductrices. *C. R. Acad. Sci. Paris Sér. I Math.*, 313(5):231–236, 1991.

[7] M. Artola and M. Cessenat. Scattering of an electromagnetic wave by a slender composite slab in contact with a thick perfect conductor. ii. inclusions (or coated material) with high conductivity and high permeability. *C. R. Acad. Sci. Paris Sér. I Math.*, 313(6):381–385, 1991.

[8] N. Bakhvalov and G. Panasenko. *Homogenisation: averaging processes in periodic media*, volume 36 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1989. Mathematical problems in the mechanics of composite materials, Translated from the Russian by D. Leĭtes.

[9] A. Bendali and K. Lemrabet. The effect of a thin coating on the scattering of a time-harmonic wave for the Helmholtz equation. *SIAM J. Appl. Math.*, 56(6):1664–1693, 1996.

[10] Alain Bensoussan, Jacques-Louis Lions, and George Papanicolaou. *Asymptotic analysis for periodic structures*, volume 5 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1978.

[11] F. Collino. High order absorbing boundary conditions for wave propagation models: straight line boundary and corner cases. In *Second International Conference on Mathematical and Numerical Aspects of Wave Propagation (Newark, DE, 1993)*, pages 161–171. SIAM, Philadelphia, PA, 1993.

[12] D. Colton and R. Kress. *Integral Equation Methods in Scattering Theory*. Krieger Pub Co, 1991.

[13] D. Colton and R. Kress. *Inverse acoustic and electromagnetic scattering theory*, volume 93 of *Applied Mathematical Sciences*. Springer-Verlag, Berlin, second edition, 1998.

[14] M. Duruflé, H. Haddar, and P. Joly. Higher order generalized impedance boundary conditions in electromagnetic scattering problems. *C.R. Physique*, 7(5):533–542, 2006.

[15] B. Engquist and A. Majda. Absorbing boundary conditions for the numerical simulation of waves. *Math. Comp.*, 31(139):629–651, 1977.

[16] B. Engquist and J.-C. Nédélec. Effective boundary conditions for acoustic and electromagnetic scattering in thin layers. Technical Report 278, Ecole Polytechnique-CMAP (France), 1993.

[17] H. Haddar and P. Joly. Stability of thin layer approximation of electromagnetic waves scattering by linear and nonlinear coatings. *J. Comput. Appl. Math.*, 143(2):201–236, 2002.

[18] H. Haddar, P. Joly, and H.-M. Nguyen. Generalized impedance boundary conditions for scattering by strongly absorbing obstacles: the scalar case. *Math. Models Methods Appl. Sci.*, 15(8):1273–1300, 2005.

[19] Houssem Haddar, Patrick Joly, and Hoai Minh Nguyen. Construction and analysis of approximate models for electromagnetic scattering from imperfectly conducting scatterers. Research Report 6302, INRIA, 09 2007.

[20] D.J. Hoppe and Y. Rahmat-Sami. *Impedance boundary conditions in electromagnetics*. Taylor & Francis, cop., 1995.

[21] J. Liu and J.M. Jin. A novel hybridization of higher order finite element and boundary integral methods for electromagnetic scattering and radiation problems. *IEEE Trans. Antennas and Propagation*, 49(12):1794–1806, 2001.

[22] Vladimir A. Marchenko and Evgueni Ya. Khruslov. *Homogenization of partial differential equations*, volume 46 of *Progress in Mathematical Physics*. Birkhäuser Boston Inc., Boston, MA, 2006. Translated from the 2005 Russian original by M. Goncharenko and D. Shepelsky.

[23] V. Mazia, S. Nazarov, and B. Plamenevskii. *Asymptotic Theory of Elliptic Boundary Value Problems in Singularly Perturbed Domains*. Springer Verlag, 2001. Translated from the Russian by Georg Heinig and Christian Posthoff.

[24] P. Monk. *Finite element methods for Maxwell's equations*. Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 2003.

[25] J.C. Nédélec. *Acoustic and electromagnetic equations*. Applied Mathematical Sciences. Springer-Verlag, Berlin, 2001.

[26] S.M. Rytov. Calcul du skin-effet par la méthode des perturbations. *J. Phys. USSR*, 2:233–242, 1940.

[27] Enrique Sánchez-Palencia. *Nonhomogeneous media and vibration theory*, volume 127 of *Lecture Notes in Physics*. Springer-Verlag, Berlin, 1980.

[28] T.B.A. Senior and J.L. Volakis. *Approximate boundary conditions in electromagnetics*. 41. IEE Electromagnetic waves series, 1995.

[29] G.N. Watson. *A Treatise on the Theory of Bessel Functions*. Cambridge University Press, second edition, 1995.