Three hours on Set Optimization

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First hour: The complete-lattice approach to set optimization. By means of four examples (duality in vector optimization, games with multi-dimensional payoffs, quantiles and risk evaluation for multi-variate random variables, multi-utility representation of non-complete preferences), the need for a new theory is made evident which is required to produce results comparable to the one-dimensional "scalar" case and match the demands of applications. The examples are shown to share a common structure which leads to complete lattices of sets as natural "image spaces" for set optimization problems. A general scheme for introducing such lattices of sets is described and a solution concept for set optimization problems is given: It has the surprising feature that attainment of the infimum and minimality are no longer equivalent concepts.

Second hour: Set-valued convex analysis. Legendre-Fenchel conjugates for set-valued convex functions are introduced and a duality theory is developed which shares all features with the corresponding "scalar" theory–which already is a new feature of the set optimization approach. The Fenchel-Moreau (biconjugation) theorem is proven as well as duality theorems of Lagrange and Fenchel-Rockafellar type are looked at. Directional derivatives and subdifferentials of set-valued convex functions are defined and corresponding optimality conditions discussed. The surprising feature is that this duality theory produces exactly the type of dual problems and variables which have previously been obtained ad hoc e.g. in mathematical finance with a clear-cut financial interpretation–but they were missing in vector and multi-criteria optimization.

Third hour: Two case studies. Two-person, zero-sum games with multi-dimensional payoffs and quantiles for multi-variate random variables are discussed in detail, and it is shown how the complete-lattice approach can–surprisingly–answer long-standing open questions. Time permitting, a more general type of set-valued convex functions, namely "translative" ones, will be discussed as a standard example for the theory presented in the second hour and as a common roof for many applications.

A reference. A.H. Hamel, F. Heyde, A. Löhne, B. Rudloff, C. Schrage, Set Optimization–a Rather Short Introduction, in: *Set Optimization and Applications – The State of the Art*, Hamel, Heyde, Löhne, Rudloff, Schrage (eds.), Springer-Verlag, Proceedings in Mathematics and Statistics 151, pp. 65-141, 2015